Physical Incompleteness from Causal Geometry in Semiclassical Gravity

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Abstract

We establish an incompleteness result for semiclassical theories of gravity that is structural, not epistemological: it arises from spacetime causal structure rather than from limitations in measurement technology, computational resources, or observer capabilities. Focusing on black hole spacetimes, we show—by explicit construction under consistency and soundness assumptions—that physically realizable quantum states exist whose complete descriptions are not derivable within a semiclassical framework restricted to a fixed exterior causal domain.

The argument rests on three elements: (i) event horizons induce causal barriers that generate observational indistinguishability classes of physical states; (ii) the Bekenstein–Hawking entropy bound provides sufficient encoding capacity to implement physical self-reference; and (iii) semiclassical gravity supports Gödel-type diagonal constructions via an arithmetic interpretation in Fock space. These are unified in a single Master Incompleteness Theorem (Theorem 4.14) with explicit hypotheses and logically independent conclusions covering geometric underdetermination, Gödelian undecidability, and their isomorphism as specific Lawvere fixed-point instances via the explicit, computable, invertible code translation induced by the triadic correspondence (Theorem 8.3).

For a solar-mass black hole with interior encoding using n_q quanta, the fractional backreaction satisfies $\delta g/g \lesssim n_q \times 10^{-76}$, ensuring semiclassical self-consistency. We discuss implications for the black hole information problem, clarify the relation to existing developments in algebraic QFT and Page-curve constructions, and identify structural constraints that any completeness-restoring quantum gravity theory must satisfy.

Keywords: Gödel incompleteness; Lawvere fixed-point theorem; black holes; causal structure; semiclassical gravity; information paradox

1 Introduction

1.1 The Central Question

Can a physical theory be fundamentally incomplete in the same rigorous sense that Gödel proved formal systems are incomplete? This question has haunted the foundations of physics since Gödel's 1931 proof that any consistent, recursively axiomatizable formal system capable of expressing basic arithmetic must contain true but unprovable statements [1].

We answer this question affirmatively for semiclassical gravity: yes, physical theories can exhibit structural incompleteness, but with geometric rather than purely logical origins. The mechanism is fundamentally different from Gödel's purely syntactic construction—it arises from the causal structure of spacetime itself.

The standard view holds that physical incompleteness, if it exists at all, is merely epistemic—a limitation of our knowledge or measurement capabilities, not a structural feature of nature itself. This paper challenges that view. We establish that semiclassical gravity—general relativity coupled with quantum field theory on curved spacetime—exhibits a form of incompleteness that is structural, enforced by spacetime geometry, and exhibits patterns parallel to Gödel's incompleteness theorems.

The key insight is that event horizons create absolute causal barriers, preventing physical information from escaping certain spacetime regions. This causal inaccessibility translates directly into theoretical incompleteness: states in causally inaccessible regions cannot be fully described by theories formulated in the accessible domain. Within the standard framework of semiclassical gravity—quantum field theory on a classical curved background, with exterior observables organized in a local algebra in the sense of algebraic QFT [46, 37]—this incompleteness manifests as a structural non-injectivity of the restriction map from global states to exterior data, formalized so as to admit Gödel-type diagonalization via the arithmetic structure of Fock space (Section 2.6, Appendix E). The framework is therefore not an external imposition of logical machinery onto physics; it is a recognition that the standard semiclassical formalism already carries the categorical resources required for the diagonal construction.

Throughout this work, "incompleteness" refers to *on-*

tological descriptive incompleteness: the inability of a theory to uniquely and exhaustively specify all physically realizable states within its own domain of applicability. This notion is *resolution- and domain-relative*: it depends on a fixed finite coding level N (Definition 2.2) and on the chosen causal domain D_O. All incompleteness results in this paper are therefore to be understood with respect to a fixed resolution N and exterior domain D_O.

1.2 Historical Context and Motivation

Gödel's incompleteness theorems demonstrated fundamental limits to formal axiomatic systems. Several researchers have explored potential connections to physics, but prior approaches either applied Gödel's results metaphorically without rigorous physical grounding, or focused on computational complexity rather than ontological incompleteness.

It is important to acknowledge that the *non-injectivity* of the restriction map $\omega \mapsto \omega|_{\mathrm{Obs}_{D_O}}$—the fact that distinct global states can agree on all observables accessible from a causally restricted domain—is well established in algebraic quantum field theory. The split property for local algebras [47], together with standard locality and microcausality assumptions [46, 37], implies that in spacetimes with causal barriers the exterior observable algebra underdetermines the global state. This phenomenon has been understood as a *technical consequence of locality* for several decades (see also [52, 53]).

What has *not* been established is the deeper structural character of this underdetermination. Our contribution is to show that the non-injectivity is not merely a technical feature of algebraic QFT but constitutes *an intrinsic Gödelian fixed-point phenomenon*: the semiclassical framework on curved spacetime *natively* carries arithmetic structure (via Fock-space number operators), self-referential encoding capacity (via the Bekenstein–Hawking entropy bound), and domain-restricted provability (via event horizons). These are not interpretive overlays imposed on the physics; they are structural features of the theory itself. The combination produces diagonal fixed points in the precise categorical sense of Lawvere [51], establishing that the underdetermination *is* a Gödelian phenomenon—not by analogy, but by demonstrated isomorphism of the underlying fixed-point architecture (Proposition 8.3).

Past attempts to connect physics and incompleteness fall into three categories.

Type 1—Metaphorical analogies. Heuristic comparisons between Gödel's theorem and various forms of physical incompleteness (measurement, computation, foundational descriptions) have appeared in the literature without a rigorous mapping at the level of formal structure. Such analogies are suggestive but do not establish a logical implication between the physical and the Gödelian phenomena.

Type 2—Computational complexity. Results showing that undecidable problems exist in physics (e.g., [20]) establish incompleteness from computational limits, not from physical structure.

Type 3—Structural correspondence (our approach). We construct an explicit correspondence $\mathcal{I}$: Geometry $\leftrightarrow$ Logic with rigorous mathematical formulation, physical realizability through explicit states, and essential uniqueness up to the stated equivalences (in the sense made precise in §2.8 below). Unlike Types 1–2 and unlike the purely algebraic observation of non-injectivity, we demonstrate that the underdetermination carries the *specific* structure of a diagonal fixed-point obstruction: not arbitrary non-uniqueness, but self-referential non-uniqueness enforced by causal geometry.

1.3 Technical Approach

We develop four innovations to establish the connection between causal geometry and incompleteness.

Gauge-invariant Gödel numbering. Classical Gödel numbering assigns natural numbers to formal statements. In quantum field theory, gauge transformations relate physically equivalent descriptions. We construct an encoding scheme that assigns unique natural numbers to physically distinct quantum states while respecting gauge symmetry, enabling arithmetic interpretation of physical theories.

Physical diagonal lemma. Gödel's diagonal lemma constructs self-referential statements through arithmetic encoding. We adapt this to curved spacetime by exploiting black hole interiors as physical encoding spaces. A black hole with Bekenstein–Hawking entropy S_{BH} can encode Gödel numbers up to $\exp(S_{\mathrm{BH}}/k_{\mathrm{B}})$, providing sufficient capacity for self-referential constructions even for modest astrophysical black holes.

Causal-domain-relative describability. Unlike Gödel's absolute unprovability, physical incompleteness is observer-relative. Different spacetime regions are describable from different causal domains, analogous to frame-dependent simultaneity in special relativity. This relativity is geometric—determined by spacetime structure—not epistemic.

Triadic correspondence and the Lawvere fixed point. Beyond these three innovations, we establish a fundamental three-way correspondence between geometry, physics, and logic. This triadic correspondence consists of structure-preserving maps F (locally covariant quantization), G (Gödel encoding), and H (geometric realization) that form a commutative diagram on the physically realized subcategory. The correspondence ensures that incompleteness in any domain—logical undecidability, physical indistinguishability, or geometric inaccessibility—implies incompleteness in the other two. We further show that both Gödel's syntactic self-reference and our geometric self-reference are instances of Lawvere's categorical fixed-point theorem [51], establishing that the connection is not merely a structural parallel but constitutes an isomorphism of the specific Lawvere fixed-point instances (via the explicit, computable, invertible code translation σ induced by the triadic correspondence — see Theorem 8.3 and Remark 8.6). The Gödelian and geometric diagonal constructions share identical categorical structure at the level of the diagonal argument, connected by an explicit computable code translation. This isomorphism operates at the level of the specific diagonal constructions, not at the level of a global equivalence of categories.

1.4 Main Results

We establish three principal results demonstrating structural incompleteness in semiclassical gravity.

Main Theorem A (Geometric indistinguishability). Let $(\mathcal{M}, g)$ be a Schwarzschild spacetime with $M \gg m_P$, and assume the split property holds for the buffered exterior/interior algebra pair $(\mathfrak{A}_{\mathrm{ext}}, \mathfrak{A}_{\mathrm{int}})$ (Hypothesis (H2) of Theorem 4.14; cf. Lemma B.1). Then for external observers at future null infinity $\mathscr{I}^+$, the restriction map $\omega \mapsto \omega|_{\mathrm{Obs}_{D_O}}$ is non-injective on the admissible state class, and there exist admissible interior states s^* whose observational indistinguishability classes contain a continuously parameterized family of physically distinct extensions (cardinality at least $2^{\aleph_0}$). Formally, for such s^* and exterior observational domain D_O:

$$\left| [s^*]_{D_O} \right| \geq 2^{\aleph_0}, \tag{1}$$

where

$$[s^*]_{D_O} := \left\{ s' \in \mathcal{S}_{\mathrm{adm}} : s'\big|_{\mathrm{Obs}_{D_O}} = s^*\big|_{\mathrm{Obs}_{D_O}} \right\}.$$

This theorem establishes that, under the stated split-property hypothesis, exterior measurements cannot distinguish a continuum of admissible interior states—not due to technological limitations, but due to causal disconnection enforced by spacetime geometry. The proof and the precise role of the admissibility class (closed under convex mixtures or, alternatively, parameterized by coherent excitations; see Theorem 3.6) appear in Section 4.

Main Theorem B (Theoretical incompleteness). Let Φ_{ext} be the restriction of semiclassical theory Φ_{semi} to statements derivable from exterior observations (equivalently, $\Phi_{\mathrm{ext}} \equiv \Phi_{D_O}$ with $D_O = \mathrm{DOC}$; we use both notations interchangeably throughout). Then Φ_{ext} is incomplete:

$$\exists\, s^* \in \mathcal{S}_{\mathrm{adm}} \text{ supported in } R_{\mathrm{int}} : \ \Phi_{\mathrm{ext}} \nvdash \mathrm{Desc}(\Phi_{\mathrm{semi}}, s^*). \tag{2}$$

The theory cannot derive complete descriptions of interior states from exterior data, exhibiting the same pattern of undecidability as Gödel's theorems (under explicit consistency and Σ_1-soundness conditions; see Theorem 4.14 and Assumptions A–B of §4.3).

Main Theorem C (Structural equivalence). Within the framework of semiclassical gravity, geometric indistinguishability arising from causal barriers is equivalent to causal-geometric incompleteness:

$$\text{Geometric Indistinguishability} \iff \text{Causal-Geometric Incompleteness}. \tag{3}$$

This equivalence establishes that the limitation is structural—enforced by spacetime geometry—rather than merely epistemological. The causal structure of spacetime determines logical properties of the theory.

Theorems A–C are consolidated in the Master Incompleteness Theorem (Theorem 4.14, §4.12), which collects the four hypotheses (recursive axiomatizability with arithmetic interpretation; split property; consistency; Σ_1-soundness) and states the joint conclusion with the logical role of each hypothesis made explicit. Individual Theorems A–C are extracted from the Master Theorem by isolating its respective conclusions.

These results establish that causal geometry instantiates Gödelian patterns of self-reference within semiclassical theories.

Our results establish a structural correspondence grounded in an explicit mathematical framework. The triadic correspondence is not a heuristic device but a rigorous structure satisfying bijectivity (each geometric configuration corresponds to a unique logical statement at the chosen resolution), structure preservation (causal relations map to provability relations), and essential uniqueness up to the stated equivalences (diffeomorphism, gauge, and recursive equivalence). Moreover, via Lawvere's fixed-point theorem, the diagonal constructions underlying both Gödel's theorem and our physical result are shown to be isomorphic as specific Lawvere fixed-point instances (via the explicit, computable, invertible code translation σ induced by the triadic correspondence — Theorem 8.3). This elevates the connection from a structural parallel to a precise isomorphism at the level of the diagonal argument, connected by a computable code translation through the triadic correspondence. The isomorphism operates at the level of the specific diagonal constructions, not at the level of a global equivalence of categories.

1.5 Scope and Implications

Our results apply specifically to semiclassical gravity (quantized matter on a classical spacetime background), the framework underlying Hawking radiation and black hole thermodynamics. The incompleteness does not necessarily extend to quantum gravity theories that transcend the semiclassical approximation.

Indeed, our theorem provides a structural constraint on any candidate completion of semiclassical gravity: any unitarizing or otherwise injectivity-restoring extension must violate at least one of the three structural ingredients identified by the Master Theorem (causal locality, sufficient interior state-space cardinality, or arithmetic representability; cf. §9.4). This identifies, in advance and in framework-independent terms, the structural cost of completion. The result also reframes the black hole information paradox: information loss is not merely a calculational puzzle but reflects structural incompleteness of semiclassical descriptions.

We emphasize what our results do *not* claim. We do not claim that Gödel's first incompleteness theorem applies directly to physics without additional representability assumptions. We claim that semiclassical gravity, when formalized with a domain-restricted provability notion and a coding interface, can instantiate Gödel-type diagonal fixed points and undecidability. We do not claim that all physical theories are incomplete. We do not claim information destruction as a theorem. And we do not claim observer-independence of completeness; rather, completeness is domain-indexed by the objective causal structure and the corresponding observable algebra. A detailed list of clarifications appears in Section 5.8.

1.6 Structure

Section 2 develops the mathematical framework including gauge-invariant Gödel numbering, the formal language for domain-restricted physical theories, and the triadic correspondence. Section 3 establishes the causal structure of black holes and physical realizability conditions. Section 4 constructs undecidable systems via the physical diagonal lemma and proves the main theorems. Section 5 addresses potential objections including holographic quantum gravity and observer-dependence. Section 6 generalizes results to rotating and charged black holes, cosmological horizons, and analog systems. Section 7 discusses implications for the information paradox and quantum gravity programs. Section 8 provides ontological interpretation distinguishing geometric from epistemic incompleteness and establishes the Lawvere fixed-point theorem as the categorical identity underlying both Gödelian and geometric diagonal arguments. Section 9 provides physical interpretation of the results, situates them within the existing literature, and specifies the conditions under which the framework's applicability could be falsified. Section 10 concludes. Appendices A–G supply explicit calculations, supporting lemmas, and technical details.

1.7 On the Nature of Physical Theorems

Physical theories occupy a unique position between pure mathematics and empirical science. This dual character is essential for understanding the status of our main result.

The mathematical claim (Sections 2–4). If a theory Φ satisfies (i) interpretation of Robinson Arithmetic Q, (ii) operation from a causally limited observational domain D_O, and (iii) description of spacetime with absolute causal barriers, then Φ exhibits geometric incompleteness in the sense of our main theorems. This is a conditional mathematical theorem whose validity depends on rigorous proof of the logical implication, not on whether such theories actually exist in nature.

The physical claim (Sections 3, 6, 9). Semiclassical gravity Φ_{semi} satisfies conditions (i)–(iii): it interprets arithmetic via Fock space, observers have causally limited domains due to event horizons, and black holes create absolute causal barriers. The applicability of the conditional theorem to Φ_{semi} therefore rests on three independently supported background assumptions: (i) general relativity, supported by gravitational-wave observations and precision tests in the strong-field regime; (ii) the existence of event horizons, supported by the singularity theorems of Penrose [48] and by direct imaging of supermassive compact objects; and (iii) the validity of semiclassical QFT in the relevant regime. The hypotheses (H1)–(H4) of the Master Theorem are not special assumptions imposed on semiclassical gravity from outside; they are conditions satisfied by any adequate formalization of the theory. A formalization that violates (H1) (recursive axiomatizability with arithmetic interpretation) either admits no computational procedure for deriving predictions, or lacks the arithmetic required to state numerical relations—both deficiencies that disqualify it as a physical theory. Hypotheses (H2) (split property), (H3) (consistency), and (H4) (Σ_1-soundness) are each established properties of the semiclassical framework or minimal sanity conditions on any sound theory. The incompleteness result is therefore not conditional on a choice among formalizations, but on the universal structural requirements that any adequate formalization must satisfy. The precise sense in which the evaluation map is *forced* by (H1)—rather than constructed—is established in Appendix E.6.

It is crucial to distinguish our result from mere limitations of knowledge. *Structural* limits concern the formal capacity of a theory to determine its model uniquely—such as Gödel's theorems showing that formal systems satisfying the standard hypotheses cannot prove all truths expressible in their language. Our incompleteness is structural in this sense: it depends only on the causal structure of spacetime and the logical structure of the theory, not on measurement precision, computational power, or observer capabilities.

This parallels the situation in special relativity, where simultaneity has no frame-independent meaning not because of measurement limitations but because of spacetime structure itself. Analogously, "state s is describable" has no causal-domain-independent meaning in our framework.

Clarification on the equivalence theorem. Our main result states an equivalence between geometric and theoretical properties:

$$\text{Geometric Indistinguishability} \iff \text{Causal-Geometric Incompleteness}.$$

The forward direction is established by the Bridge Theorem (Theorem 2.12), and the reverse direction follows from locality of quantum field theory. The proof does not rely on analogy or metaphor.

2 Mathematical Framework

This section introduces the formal machinery used in the incompleteness argument. The goal is to make precise (i) a gauge-invariant encoding of physically distinct states by natural numbers (§2.2–2.4), (ii) basic cardinality statements about the relevant state spaces and equivalence classes (§2.5), (iii) an arithmetic interpretation sufficient to support Gödel-type diagonal constructions within the semiclassical setting (§2.6), and (iv) the structural correspondence linking causal geometry, domain-relative observability, and logical undecidability (§2.7–2.8).

Throughout, we work in the semiclassical regime: a classical spacetime geometry (M, g) coupled to quantum fields on (M, g), with observables understood in the standard algebraic/QFT-in-curved-spacetime sense and expectation values taken in an admissible state class (specified later; e.g., Hadamard states when renormalized stress-energy is invoked). The formal system and encoding apparatus introduced here are not intended to model all of physics; they are constructed to isolate the structural features relevant to causal-domain-restricted describability.

2.1 Table of Symbols (Selected)

Spacetime and causal structure.

(M, g)	spacetime manifold with Lorentzian metric
$\mathscr{I}^+$	future null infinity (asymptotically flat setting)
$J^{\pm}(p)$	causal future/past of point p
$D(S)$	domain of dependence of set S
R_{ext}, R_{int}	exterior/interior regions (relative to a horizon)
$\mathcal{H}$	event horizon, $\mathcal{H} = \partial J^-(\mathscr{I}^+)$
r_s	Schwarzschild radius, $r_s = 2GM/c^2$

Observers and causal domains.

O	observer (typically exterior/asymptotic)
D_O	observational (causal) domain associated with O
Obs_O	algebra of observables supported in D_O

Semiclassical / QFT data.

Φ_{semi}	semiclassical framework (QFT on curved spacetime)
$\lvert\psi\rangle$	quantum state vector
ω	state (functional) on the observable algebra
$\langle\varphi\rangle_\psi$	expectation value of observable φ in state $\lvert\psi\rangle$
$\langle T_{\mu\nu}\rangle^{\text{ren}}$	renormalized stress-energy expectation value
$[\psi]$	gauge-equivalence class of $\lvert\psi\rangle$

Logical / formal apparatus.

$\mathcal{L}$	formal language of the theory
$\ulcorner\varphi\urcorner$	Gödel code of formula φ
$\vdash$	provability relation
$\models$	semantic truth/satisfaction relation
$\text{Desc}(\Phi, s)$	"complete description" predicate for state s in theory Φ

Notation convention for states. Throughout we write s for a state, equivalently understood as either an algebraic state functional ω on the observable algebra or, in a Hilbert-space picture, a vector $\lvert\psi\rangle$ (or, more generally, a density operator). The class of physically realizable states is denoted $\mathcal{S}_{\text{adm}}$ (Definition 3.5). We use ω in preference to s only where explicit reference to the algebraic functional structure is required (e.g., when discussing restriction maps to subalgebras in the Master Theorem of §4.12).

Encoding maps and information-theoretic quantities.

γ	gauge-invariant encoding map (Gödel numbering)
$\gamma([\psi]) \in \mathbb{N}$	natural number assigned to state class $[\psi]$
S_{BH}	Bekenstein–Hawking entropy
k_B, $\hbar$, c, G	Boltzmann constant, Planck constant, speed of light, Newton constant

2.2 Formal Language for Domain-Restricted Physical Theories

For the Gödel-type constructions used later, we represent a suitably chosen fragment of semiclassical gravity in a formal language. The aim is not to capture all of semiclassical physics in full detail, but to specify a recursively axiomatizable framework rich enough to (i) express causal-domain restriction and observational access, (ii) talk about physically realizable states and their descriptive content, and (iii) support an arithmetic interpretation used in diagonal arguments.

Language $\mathcal{L}$ (many-sorted first-order). We use a many-sorted first-order language with sorts for: (i) events/points, (ii) regions, (iii) physical states, and (iv) natural numbers.

Non-logical symbols (schematic; fixed once and used throughout):

- *Causal structure:* a binary relation $\preceq$ on events (causal precedence), and predicates $J^+(p, q)$, $J^-(p, q)$ expressing causal reachability.
- *Horizon/domain data:* unary predicates $\text{InExt}(p)$ and $\text{InInt}(p)$ (exterior/interior membership), and a predicate $\text{InDom}(p, O)$ specifying the observational domain D_O.
- *Observational access:* $\text{Obs}(\varphi, O)$ meaning "the observable φ is supported in and accessible from D_O."
- *Realizability:* $\text{Real}(s)$ meaning "s is physically realizable" (within the admissible state class of the semiclassical regime).
- *Expectation/evaluation:* a ternary relation $\text{Eval}(s, \varphi, q)$ meaning "the expectation value of φ in state s equals q," where q ranges over a coded set of rationals.

- *Description predicate:* $\mathrm{Desc}(\Phi, s, n)$ meaning "n codes a complete description of state s within theory Φ"—operationally, n encodes a tuple of $\mathrm{Eval}(s, A_i, q_i)$ relations for a separating family of observables $\{A_i\}$ at the coding resolution; formalized in Section 2.10; the role of 'completeness' is domain-relative.

Arithmetic vocabulary. The language includes the symbols required to interpret arithmetic on the $\mathbb{N}$-sort (at minimum those of Robinson Arithmetic Q, introduced explicitly in Section 2.6).

Theory schema Φ_{semi} (effective axiomatization). We work with an effectively axiomatized theory schema intended to formalize the semiclassical regime relevant to our constructions. At the level needed for the incompleteness argument, Φ_{semi} includes axioms expressing:

(A) Semiclassical dynamics (schematic):

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4}\, \langle T_{\mu\nu}\rangle^{\mathrm{ren}}\,, \tag{4}$$

together with the conservation condition $\nabla^\mu \langle T_{\mu\nu}\rangle^{\mathrm{ren}} = 0$.

(B) QFT locality and causality: Local commutativity (microcausality) for observables supported in spacelike-separated regions, and standard locality assumptions ensuring that domain-supported observables depend only on the physics in the causal domain.

(C) Admissible state class and renormalization regularity: When $\langle T_{\mu\nu}\rangle^{\mathrm{ren}}$ is used, states are restricted to an admissible class (e.g., Hadamard) for which renormalized expectation values are well-defined.

(D) Domain restriction / observational content: Axioms relating $\mathrm{Obs}(\varphi, O)$ to support in D_O and ensuring that statements in the exterior restriction Φ_{ext} are grounded solely in Obs_O.

(E) Gauge covariance/invariance principles: Axioms specifying the action of gauge redundancies on descriptions and the invariance of physical content under that action.

Remarks on effective axiomatizability. For Gödel-type results, we require that the deductive apparatus is recursive and that the axioms are recursively enumerable. We therefore work with an explicitly defined recursively axiomatizable fragment $\Phi_0 \subseteq \Phi_{\mathrm{semi}}$ sufficient for the constructions in Sections 2–4. Concretely, Φ_0 consists of:

(i) A finite set of axiom schemas for the semiclassical Einstein equation (4) restricted to the Schwarzschild background (fixing M and the background geometry eliminates the need for the full nonlinear dynamics).

(ii) The axioms of Robinson Arithmetic Q interpreted via the Fock-space operators $\hat{V}$, $\hat{N}_+$, $\hat{N}_\times$ (Appendix E), together with finitely many axioms specifying eigenvalue relations on the number-state basis.

(iii) Finitely many axioms encoding the domain-restriction predicates: $\mathrm{InExt}(p)$, $\mathrm{InInt}(p)$, $\mathrm{Obs}(\varphi, O)$, and the causal structure axioms for the Schwarzschild exterior (specifically, that $R_{\mathrm{int}} \cap J^-(D_O) = \varnothing$).

(iv) The algebraic axioms for the exterior observable algebra $\mathfrak{A}(D_O)$: $*$-algebra structure, microcausality, and the restriction/extension properties used in the Bridge Theorem.

(v) A recursively enumerable set of instances of the coding-interface predicates Ad_N, $\mathrm{SameExt}_N$, and $\mathrm{Desc}_{D_O,N}$ (which are Δ_0 and hence representable in any extension of Q).

Each component is either finite or recursively enumerable, so Φ_0 is recursively axiomatizable. By construction, Φ_0 interprets Q and contains all predicates required for the diagonal lemma and the domain-restricted describability statements. The incompleteness results of Section 4 apply to Φ_0 (and *a fortiori* to any consistent extension).

Truth versus provability. Well-formed formulas $\varphi \in \mathcal{L}$ are interpreted in intended semiclassical models $(M, g, \ldots)$ satisfying Φ_{semi}. Provability ($\Phi_{\mathrm{semi}} \vdash \varphi$) is distinct from semantic truth ($\mathcal{M} \models \varphi$).

2.3 Physical States and Gauge Equivalence

State space. Rather than assuming a single global Hilbert space in full generality, we allow a standard QFT-in-curved-spacetime/algebraic formulation. Let $\mathfrak{A}(M, g)$ denote the algebra of observables associated with the background (M, g), and let $\mathcal{S}(M, g)$ denote a chosen admissible class of states (e.g., Hadamard states when renormalized quantities are used). When a Hilbert-space picture is available, one may represent states by GNS vectors $|\psi\rangle$ (or, more generally, by density operators in the GNS representation); the formal developments below do not depend on choosing a particular representation.

Gauge action. Let $\mathcal{G}$ be the relevant gauge group (internal gauge symmetries, and—where appropriate—diffeomorphism covariance treated as a redundancy of description). The action of $g \in \mathcal{G}$ is represented as an automorphism $\alpha_g \colon \mathfrak{A} \to \mathfrak{A}$ on observables. States transform by pullback:

$$s \;\mapsto\; s^g\,, \qquad s^g(A) := s\big(\alpha_g(A)\big) \quad \text{for all } A \in \mathfrak{A}\,. \tag{5}$$

Gauge equivalence. Two states s and s' are gauge-equivalent (written $s \sim_g s'$) if there exists $g \in \mathcal{G}$ such that $s' = s^g$. The corresponding equivalence class is denoted $[s]_g$.

Gauge-invariant observables and physical distinguishability. An observable $A \in \mathfrak{A}$ is gauge-invariant if $\alpha_g(A) = A$ for all $g \in \mathcal{G}$. Gauge-equivalent states agree on all gauge-invariant observables:

$$s \sim_g s' \implies s(A) = s'(A) \quad \text{for all gauge-invariant } A\,.$$

Conversely, physical distinguishability (in the gauge sense) is defined relative to the gauge-invariant subalgebra: two gauge-inequivalent classes $[s_1]_g \neq [s_2]_g$ are physically distinct if and only if there exists a gauge-invariant observable A such that $s_1(A) \neq s_2(A)$.

Separation from observational equivalence. Gauge equivalence is conceptually distinct from the domain-relative observational equivalence introduced later ($s \sim_O s'$ meaning agreement on all observables supported in D_O). In the main argument, we quotient by gauge redundancies first (to ensure physical coding), and only then consider the additional coarse-graining induced by restriction to a causal domain.

2.4 Gauge-Invariant Gödel Numbering

Challenge. Classical Gödel numbering assigns natural numbers to syntactic objects. In a physical setting, the relevant objects are not arbitrary vectors in a Hilbert space but physically meaningful state descriptions modulo gauge redundancies. Moreover, the totality of physically admissible states in QFT on curved spacetime is typically uncountable; therefore, an injective map from all such states into $\mathbb{N}$ cannot exist without an explicit restriction. Our strategy is to define a gauge-invariant, effectively describable equivalence relation that captures the physically distinguishable content at a fixed resolution, and to Gödel-encode the resulting (countable) space of finite descriptions.

Definition 2.1 (Gauge equivalence)**.** Let $\mathcal{G}$ be the relevant gauge group acting on the algebra of observables via automorphisms α_g. Two states s, s' are gauge-equivalent (written $s \sim_g s'$) if there exists $g \in \mathcal{G}$ such that $s' = s^g$, where $s^g(A) := s(\alpha_g(A))$ for all observables A. The corresponding equivalence class is denoted $[s]_g$.

Definition 2.2 (Coding data and finite-resolution physical equivalence)**.** Fix a countable family $\{O_i\}_{i\in\mathbb{N}}$ of gauge-invariant observables that is *separating* for the admissible state class—i.e., for any two distinct admissible states $s \neq s'$ there exists i with $\langle O_i\rangle_s \neq \langle O_i\rangle_{s'}$—and dense in the weak-$*$ topology on the gauge-invariant subalgebra one wishes to encode. Fix also a resolution scheme that assigns to each i a rational approximation map

$$Q_i : \mathbb{R} \to \mathbb{Q}\,,$$

for example $Q_i(x) := \lfloor 10^i x \rfloor / 10^i$. For $N \in \mathbb{N}$, define the *finite coding datum* of a state s by

$$C_N(s) := \big(Q_1(\langle O_1\rangle_s),\ Q_2(\langle O_2\rangle_s),\ \ldots,\ Q_N(\langle O_N\rangle_s)\big) \in \mathbb{Q}^N\,.$$

Define *finite-resolution physical equivalence at level N* by

$$s \sim^N_{\text{phys}} s' \quad \Longleftrightarrow \quad C_N(s) = C_N(s')\,.$$

This relation is gauge-invariant (since the O_i are gauge-invariant) and yields a countable family of equivalence classes because $\mathbb{Q}^N$ is countable for each fixed N.

Remark 2.3. The relation $\sim^N_{\text{phys}}$ is not intended to identify all physically distinct states. It identifies states that are indistinguishable at the chosen coding resolution determined by the choice of observables $\{O_i\}$ and the rational quantization maps Q_i. This is the correct level at which Gödel-type encoding can be carried out without presupposing that the full physical state space is countable.

Definition 2.4 (Gauge-invariant Gödel numbering at level N)**.** Fix $N \in \mathbb{N}$. A map

$$\gamma_N : \big(\mathcal{S} / \sim^N_{\text{phys}}\big) \longrightarrow \mathbb{N}$$

is a *gauge-invariant Gödel numbering* (at level N) if it satisfies:

(i) *Well-definedness:* If $s \sim^N_{\text{phys}} s'$, then $\gamma_N([s]) = \gamma_N([s'])$.

(ii) *Injectivity:* $\gamma_N([s]) = \gamma_N([s'])$ implies $[s]^N_{\text{phys}} = [s']^N_{\text{phys}}$.

(iii) *Effective encodability:* There is a Turing-computable procedure that, given $C_N(s)$, outputs $\gamma_N([s])$.

Construction (standard Gödel encoding of rational tuples). Encode each rational $q = a/b$ in lowest terms ($b > 0$) via a standard injective map $\text{enc}\colon \mathbb{Q} \to \mathbb{N}$, e.g.,

$$\text{enc}(a/b) := \langle\, \text{sign}(a),\ |a|,\ b\,\rangle\,,$$

where $\langle\cdot,\cdot,\cdot\rangle$ is a fixed primitive-recursive tupling function. Then define

$$\gamma_N\big([s]^N_{\text{phys}}\big) \ := \ \prod_{i=1}^{N} p_i^{\,\text{enc}(Q_i(\langle O_i\rangle_s))}\,, \tag{6}$$

where p_i is the i-th prime. Unique factorization ensures that γ_N codes the entire finite tuple $C_N(s)$.

Proposition 2.5 (Injectivity)**.** *$\gamma_N([s]) = \gamma_N([s'])$ implies $C_N(s) = C_N(s')$, hence $[s]^N_{\text{phys}} = [s']^N_{\text{phys}}$.*

Proof. By the Fundamental Theorem of Arithmetic, the prime exponents in γ_N are uniquely determined; therefore $\text{enc}(Q_i(\langle O_i\rangle_s)) = \text{enc}(Q_i(\langle O_i\rangle_{s'}))$ for all $i \leq N$. The map $\text{enc}\colon \mathbb{Q} \to \mathbb{N}$, defined on lowest-terms representatives via the primitive-recursive tupling $\langle \text{sign}(a), |a|, b\rangle$, is injective on its domain: this follows from the uniqueness of the lowest-terms representation together with the injectivity of the tupling function. Hence $Q_i(\langle O_i\rangle_s) = Q_i(\langle O_i\rangle_{s'})$ and thus $C_N(s) = C_N(s')$. □

Proposition 2.6 (Gauge invariance)**.** *If $s \sim_g s'$, then $C_N(s) = C_N(s')$, hence $s \sim^N_{\text{phys}} s'$ and $\gamma_N([s]) = \gamma_N([s'])$.*

Proof. Each O_i is gauge-invariant, so $\langle O_i\rangle_s = \langle O_i\rangle_{s'}$ for all i; applying Q_i preserves equality. □

Proposition 2.7 (Computability)**.** *For fixed N, the map γ_N is Turing-computable from the finite coding datum $C_N(s)$.*

Proof. Computing $\text{enc}(q)$ for a rational q is primitive recursive; computing p_i for $i \leq N$ is effective; exponentiation and multiplication of integers are computable operations. □

Interpretational clarification (truth versus derivability). The Gödel numbering γ_N is a coding device. In later sections, the incompleteness claim concerns derivability within a domain-restricted theory Φ_{ext}: even if a statement about coded data (e.g., "$C_N(s) = v$") is true in the intended model, it may fail to be provable in Φ_{ext} once the theory is restricted to a causal domain that does not fix the relevant observables. This is the precise analogue of "true but unprovable" statements in Gödel's setting.

Choice of observables. In applications to black hole spacetimes, the family $\{O_i\}$ naturally includes asymptotic gauge-invariant charges (ADM mass M, angular momentum J, electric charge Q, multipole moments) as well as a countable family of smeared local gauge-invariant densities (e.g., integrals of φ^2, $(\nabla\varphi)^2$, $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ against a countable dense set of test functions). The use of smeared observables (rather than pointwise field values) is standard in QFT and ensures mathematical well-definedness.

Physical precision bound and state distinguishability. For a black hole of horizon area A, the Bekenstein–Hawking entropy satisfies

$$S_{\text{BH}}/k_{\text{B}} \;=\; \frac{A}{4\,\ell_{\text{P}}^2} \;=\; \frac{4\pi G M^2}{\hbar c} \qquad \text{(Schwarzschild)}\,, \tag{7}$$

so the number of distinguishable microstates is of order $N_{\max} \sim \exp(S_{\text{BH}}/k_{\text{B}})$. For $M = M_\odot$, $S_{\text{BH}}/k_{\text{B}} \approx 1.05 \times 10^{77}$; for $M = 10^6\,M_\odot$, $S_{\text{BH}}/k_{\text{B}} \approx 1.05 \times 10^{89}$. This bound constrains the number of physically distinguishable interior configurations available to the encoding construction. The magnitude of $\gamma_N([s])$ can vastly exceed $N_{\max}$; what matters for the later incompleteness argument is that (a) distinct physically distinguishable classes admit distinct codes (injectivity on the chosen equivalence classes) and (b) the coding is effective.

Uniqueness up to recursive equivalence. Any two acceptable Gödel numberings (γ_N, γ'_N) defined on the same class of finite coding data are recursively equivalent: there exist total computable functions f, g such that $\gamma'_N = f \circ \gamma_N$ and $\gamma_N = g \circ \gamma'_N$. Consequently, the incompleteness statements developed later are invariant under the specific choice of effective Gödel numbering, provided the numbering satisfies the stated conditions.

2.5 Countability of Physical States

Theorem 2.8 (Countability of finite-resolution physical descriptions)**.** *Fix a countable family of gauge-invariant observables $\{O_i\}_{i\in\mathbb{N}}$ and a rational quantization scheme $\{Q_i\}_{i\in\mathbb{N}}$ as in Section 2.4. For any finite $N \in \mathbb{N}$, the set of finite coding data*

$$C_N(s) := \big(\, Q_1(\langle O_1\rangle_s),\; \ldots,\; Q_N(\langle O_N\rangle_s) \,\big) \;\in\; \mathbb{Q}^N$$

is countable. Consequently, the quotient set $\mathcal{S}/\!\sim^N_{\text{phys}}$ of finite-resolution physical equivalence classes is countable, and any Gödel numbering $\gamma_N \colon (\mathcal{S}/\!\sim^N_{\text{phys}}) \to \mathbb{N}$ has countable image.

Proof. For fixed N, the map $s \mapsto C_N(s)$ takes values in $\mathbb{Q}^N$. Since $\mathbb{Q}$ is countable, the finite Cartesian product $\mathbb{Q}^N$ is countable. Therefore $\mathrm{Im}(C_N) \subseteq \mathbb{Q}^N$ is countable, and so is the set of equivalence classes $\mathcal{S}/\!\sim^N_{\text{phys}}$ induced by equality of C_N. Finally, $\mathrm{Im}(\gamma_N) \subseteq \mathbb{N}$ is countable. □

Remark 2.9 (What is and is not being claimed). The theorem does not assert that the full physical state space $\mathcal{S}$ is countable. In QFT on curved spacetime the admissible state space is typically uncountable. The point is that for any fixed finite description level (finite list of observables, finite rational resolution), the space of distinct description data becomes countable, which is the correct level for Gödel-type encoding.

Remark 2.10 (Operational stability of encoding under exterior observables). The Bekenstein–Hawking bound $N_{\max} \sim \exp(S_{\text{BH}}/k_{\text{B}})$ guarantees sufficient distinguishable microstates. Moreover, because the encoding uses only *gauge-invariant* and *exterior-accessible* observables $\{O_i\}$ (asymptotic charges + smeared local densities), the distinguishability of the coded classes $[\psi]^N_{\text{phys}}$ remains stable under the restriction map to Obs_{D_O}. The split property (Lemma B.1) ensures that distinct interior encodings produce identical exterior data, while the chosen observables remain separating at the fixed resolution N. Thus the encoding capacity is both information-theoretically and operationally adequate within the semiclassical regime.

Physical relevance: entropy bounds and distinguishability. For black holes the Bekenstein–Hawking entropy provides an independent bound on the number of mutually distinguishable interior microstates compatible with fixed exterior charges. For a Schwarzschild black hole,

$$S_{\text{BH}}/k_{\text{B}} \;=\; \frac{A}{4\,\ell_{\text{P}}^2} \;=\; \frac{4\pi G M^2}{\hbar c}\,.$$

Hence the number of distinguishable microstates scales as $N_{\text{states}} \lesssim \exp(S_{\text{BH}}/k_{\text{B}})$. For $M = M_\odot$, $S_{\text{BH}}/k_{\text{B}} \approx 1.05 \times 10^{77}$, so $N_{\text{states}} \lesssim \exp(1.05 \times 10^{77})$. For $M = 10^6\,M_\odot$, $S_{\text{BH}}/k_{\text{B}} \approx 1.05 \times 10^{89}$.

This entropy bound constrains the number of physically distinguishable configurations available to the interior encoding used in later sections. It is conceptually distinct from Theorem 2.8 (which is purely a statement about finite coding data), but it motivates the choice of a finite description level adequate to encode the required arithmetic within the semiclassical regime.

Implication. At any fixed finite resolution N, physical state descriptions (modulo $\sim^N_{\text{phys}}$) form a countable set and admit effective Gödel numbering. This suffices for the arithmetization and diagonal constructions developed below.

2.6 Arithmetic Interpretation

To connect causal-domain restriction with Gödel-type incompleteness, we require that the formalized theory can reason about (codes of) its own descriptions. Concretely, we use two standard ingredients.

(i) Arithmetization of syntax. Given that Φ is recursively axiomatized and that formulas and proofs are finite strings, one can fix a Gödel coding $\ulcorner\cdot\urcorner$ mapping formulas and proofs into natural numbers. In the metatheory, the relation "p is a valid Φ-proof of the formula with code n" is primitive recursive. For any theory extending a weak arithmetic base (such as Robinson Arithmetic Q), this proof relation can be represented by a formula $\mathrm{Proof}_\Phi(p,n)$ in the language of arithmetic, and the corresponding provability predicate can be defined by

$$\mathrm{Prov}_\Phi(n) \;:\equiv\; \exists p\ \mathrm{Proof}_\Phi(p,n)\,.$$

(ii) Representation of computable predicates on physical codes. Fix N and the physical coding γ_N from Section 2.4. A (meta-level) predicate P on finite-resolution physical classes is *arithmetically representable* (relative to γ_N) if there exists a formula $\theta_P(n)$ such that for every class $[s]^N_{\mathrm{phys}}$,

$$P\big([s]^N_{\mathrm{phys}}\big) \text{ holds } \implies \Phi \vdash \theta_P\big(\gamma_N([s]^N_{\mathrm{phys}})\big)\,,$$

and, when needed, failure of P can be expressed by Φ proving $\neg\theta_P(\gamma_N([s]))$ for the relevant instances (or by using a standard Σ_1-representation, depending on the application).

The later diagonal constructions require only the standard level of representability used in Gödel's incompleteness theorem: enough arithmetic to represent the proof predicate and to carry out the diagonal lemma.

Key consequence (diagonal lemma, classical form). Assume Φ is consistent, recursively axiomatizable, and extends Q. Then for any formula $\psi(n)$ with one free number variable, there exists a sentence G such that

$$\Phi \vdash G \;\leftrightarrow\; \psi(\ulcorner G\urcorner)\,. \tag{8}$$

This is the standard diagonal lemma [1, 33]. In later sections we apply this mechanism to predicates that encode domain-restricted describability statements, thereby producing Gödel-like "true but not derivable (within Φ_{ext})" behavior.

Remark (interface with the physical coding). The syntactic Gödel code $\ulcorner\cdot\urcorner$ applies to formulas/proofs of Φ. The physical coding γ_N applies to finite-resolution physical description data. The bridge between them is supplied by the Desc predicate introduced in Section 2.10: $\mathrm{Desc}(\Phi, s, n)$ states that n codes (at the chosen resolution) a complete description of s within Φ. The incompleteness statement will concern what Φ_{ext} can derive about Desc for physically realizable states once Φ is restricted to a fixed causal domain.

Realization of Robinson Arithmetic in Fock space. The arithmetic interpretation is not merely formal but is grounded in standard quantum field theory. We realize the signature of Robinson Arithmetic $Q = (0, S, +, \times)$ via multi-mode Fock space operators:

- *Natural numbers:* Fock states $|n\rangle$ with $n \in \mathbb{N}$, eigenvalues of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$.
- *Zero:* The vacuum state $|0\rangle$.
- *Successor:* The shift isometry $\hat{V}\,|n\rangle = |n+1\rangle$, defined by $\hat{V} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$. (The creation operator satisfies $\hat{a}^\dagger = \hat{V}\sqrt{\hat{N}+1}$; the successor function is defined at the level of the Fock-space label structure via $\hat{V}$, avoiding the state-dependent normalization factor.)
- *Addition:* On the two-mode space $\mathcal{H}_A \otimes \mathcal{H}_B$, define the total number operator $\hat{N}_+ := \hat{N}_A + \hat{N}_B$. On product number states, $\hat{N}_+(|m\rangle_A\,|n\rangle_B) = (m+n)\,|m\rangle_A\,|n\rangle_B$. The sum $m+n$ is obtained deterministically as the eigenvalue of $\hat{N}_+$.
- *Multiplication:* On the same two-mode space, define $\hat{N}_\times := \hat{N}_A\,\hat{N}_B$. Then $\hat{N}_\times(|m\rangle_A\,|n\rangle_B) = (m\cdot n)\,|m\rangle_A\,|n\rangle_B$. The product is obtained deterministically as the eigenvalue of $\hat{N}_\times$.

Remark 2.11 (Physical realizability of number eigenstates). The exact number eigenstates $|n\rangle$ used in the interpretation of Q are not required to be exact energy eigenstates of the full interacting Hamiltonian. It suffices that there exist *low-energy coherent excitations* (finite-particle states with small total occupation number $n_q \ll 1$) whose expectation values of the number operators $\hat{N}$, $\hat{N}_+$, $\hat{N}_\times$ and the shift isometry $\hat{V}$ reproduce the arithmetic relations up to arbitrarily small error. Such states are Hadamard, satisfy the quantum energy inequalities, and produce fractional backreaction $\delta g/g \lesssim n_q \times 10^{-76}$ (see Lemma B.6 and Appendix A.2). Thus mathematical representability and physical realizability are fully reconciled in the semiclassical regime.

The axioms of Q reduce to identities of eigenvalues on the number-state basis and therefore hold exactly on the intended interpretation domain. The full verification is given in Appendix E.

2.7 Structural Correspondence

Central idea. The incompleteness phenomenon developed in this paper is organized by a structural correspondence linking three kinds of objects: (i) logical undecidability in a formal arithmetic setting, (ii) domain-relative physical indistinguishability induced by causal barriers, and (iii) geometric inaccessibility encoded by causal structure.

The role of this section is not to assert a literal identity between "logic" and "physics" *tout court*, but to define a precise bridge: a mapping from suitably coded logical objects (sentences/proofs) to physically realizable encoding states, and a parallel mapping from causal-domain restriction to a provability restriction. The resulting correspondence is "structural" in the sense that it preserves the relations relevant to the diagonal construction and to domain-restricted describability.

Correspondence table (schematic).

Formal / logical side	Physical / causal-geometric side
Arithmetic theory F (recursively axiomatized)	Exterior-restricted theory Φ_{ext} (domain-restricted)
Sentence G (Gödel-type fixed point)	Interior encoding state s_G (physically realizable)
Provability predicate $\text{Prov}_F(\ulcorner G \urcorner)$	Exterior derivability of $\text{Desc}(\Phi_{\text{semi}}, s_G, n)$ in Φ_{ext}
Truth in the intended arithmetic model	Physical realizability in the intended semiclassical model
Diagonal (fixed-point) construction	Physical diagonalization via interior encoding capacity

Theorem 2.12 (Bridge Theorem)**.** *Let F be a consistent, recursively axiomatizable arithmetic theory extending Q. Let Φ_{semi} be the semiclassical framework and Φ_{ext} its restriction to a fixed exterior observational domain D_O. Assume:*

(A) there exists an effective gauge-invariant coding γ_N at finite resolution N (Definition 2.4) and a description predicate $\text{Desc}(\Phi_{\text{semi}}, s, n)$ that is Δ_0 in the language of Φ_0 (hence arithmetically representable in any extension of Q);

(B) there is a physically realizable encoding map $\iota\colon \text{Sent}(F) \to \mathcal{S}_{\text{int}}$ such that the code of a sentence is recoverable from the interior configuration at the chosen resolution.

Then there exists a bridge between the Gödel sentence G_F for F and an interior state $s_G := \iota(G_F)$ such that:

1. *If $F \vdash G_F$, then $\Phi_{\text{ext}} \vdash \text{Desc}(\Phi_{\text{semi}}, s_G, n_G)$.*
2. *If $F \vdash \neg G_F$, then $\Phi_{\text{ext}} \vdash \neg\text{Desc}(\Phi_{\text{semi}}, s_G, n_G)$.*

Moreover, under the causal restriction induced by the horizon and the split property, there exist physically realizable s_G for which neither $\text{Desc}(\Phi_{\text{semi}}, s_G, n_G)$ nor its negation is derivable in Φ_{ext}, although both are semantically true in the intended semiclassical model. This is genuine Gödel-type incompleteness (self-referential fixed point) and not mere language restriction.

Remark 2.13. The predicate $\text{Desc}(\Phi, s, n)$ is explicitly constructed in Section 2.10 as a Δ_0 formula (bounded quantifiers over the finite coding datum $C_N(s)$). Its arithmetic representability in Φ_0 follows directly from the fact that all predicates Ad_N, SameExt_N and Eval are Δ_0 (see item (v) in the axiomatization of Φ_0). Thus the diagonal lemma applies directly inside Φ_{ext} and produces a true but unprovable statement of the required form.

Remark 2.14. The correspondence is not a single bijection between "all geometry" and "all logic." What is required (and what is provided) is a structure-preserving bridge for the specific objects used in the incompleteness construction: codes, fixed-point sentences, and domain-restricted derivability statements.

Remark 2.15. The directionality "$F \vdash G \iff \Phi_{\text{ext}} \vdash \text{Desc}(\ldots)$" as a biconditional is generally too strong to assert at this stage. The correct claim is an implication pattern mediated by the chosen encoding ι and by the definition of Φ_{ext}; equivalences can be stated later only under explicit assumptions that make both directions provable.

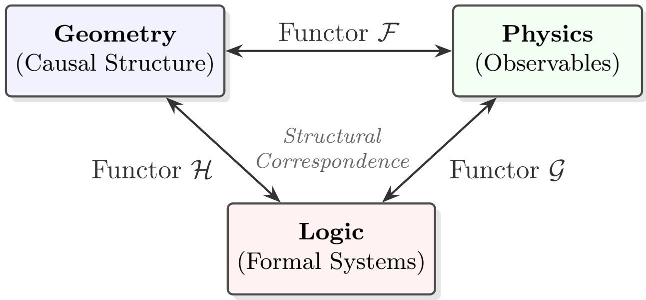

Figure 1: The Triadic Correspondence: A categorical representation of the structural mapping.

2.8 The Triadic Correspondence: Geometry, Physics, and Logic

The connection developed in this paper is not a loose analogy between "physics" and "logic," but a precise bridge between three kinds of structures: (i) causal geometry (including causal barriers), (ii) domain-relative physical observability (including indistinguishability classes), and (iii) logical derivability/undecidability under an arithmetic interpretation. We package this bridge using categorical language, with the understanding that the resulting correspondence is a controlled equivalence on a physically realized subfamily of objects, not an identification of "all geometry" with "all logic." The technical specification of the categories, their morphisms, and the auxiliary propositions are collected in Appendix G; the present subsection states only what is required for the main argument.

The three categories (in brief)

$\mathbf{Cat}_G$ **(Geometry).** Objects are domains of outer communication (DOC) of stationary, asymptotically flat, electrovac black hole spacetimes, characterized in the standard uniqueness setting by Kerr–Newman parameters (M, J, Q) up to isometry. Morphisms are causal isometric embeddings preserving asymptotic structure. Interior regions are not objects of $\mathbf{Cat}_G$.

$\mathbf{Cat}_P$ **(Physics).** Objects are locally covariant quantum field theories on globally hyperbolic spacetimes in the sense of Brunetti–Fredenhagen–Verch (BFV) [37, 46]: each (DOC, g) is assigned a unital $*$-algebra

$\mathfrak{A}(\mathrm{DOC}, g)$ satisfying isotony, covariance, microcausality, and the time-slice property. For stationary black hole exteriors, $\mathfrak{A}(\mathrm{DOC}, g)$ contains an asymptotic subalgebra $\mathfrak{A}_\infty$ generated by gauge-invariant asymptotic charges. Morphisms are injective $*$-homomorphisms α_ψ induced by causal isometric embeddings.

$\mathbf{Cat}_L$ (Logic). Objects are consistent, recursively axiomatizable formal systems $(\mathcal{L}, \vdash)$ that interpret Robinson Arithmetic Q, considered up to bi-interpretability (or, equivalently, recursive equivalence of Gödel numberings). We restrict $\mathbf{Cat}_L$ to the physically realized subcategory arising as the essential image of the Gödel-encoding functor defined below. Morphisms are bi-interpretations preserving provability in both directions.

Why a triadic bridge? A direct map "Geometry $\to$ Logic" is formally possible but physically opaque: it does not explain how a geometric feature becomes a statement about derivability, nor how the correspondence is anchored in observables. The intermediate "Physics" layer provides the operational content: causal geometry determines which observables are available in a domain, and the logical layer is built by encoding the domain-restricted descriptive content of those observables.

The functors (schematic)

- *Functor F: $\mathbf{Cat}_G \to \mathbf{Cat}_P$ (locally covariant quantization):* on objects, $F(\mathrm{DOC}, g) := \mathfrak{A}(\mathrm{DOC}, g)$ (the BFV observable algebra on the exterior region); on morphisms, local covariance provides the induced injective $*$-homomorphism. Asymptotic charges (ADM mass, angular momentum, total electric charge) are gauge-invariant elements of $\mathfrak{A}_\infty$ and serve as the asymptotic anchor of the correspondence.
- *Functor G: $\mathbf{Cat}_P \to \mathbf{Cat}_L$ (Gödel encoding of domain-restricted descriptive content):* fix a finite-resolution level N and the gauge-invariant coding scheme of §2.4; from $\big(\mathfrak{A}(\mathrm{DOC}, g), \text{coding scheme}\big)$ build a recursively axiomatizable formal system $\mathcal{L}_{\mathfrak{A},N}$ whose intended semantics is given by coded expectation values of the selected observables in admissible states. On objects, $G(\mathfrak{A}(\mathrm{DOC}, g)) := \mathcal{L}_{\mathfrak{A},N}$. The encoding is injective only on finite-resolution equivalence classes $\mathcal{S}/{\sim}^N_{\mathrm{phys}}$, and any two acceptable effective encodings at fixed N are recursively equivalent (Proposition G.3 of Appendix G).
- *Functor H: $\mathbf{Cat}_L \to \mathbf{Cat}_G$ (geometric realization on the physically realized subcategory):* defined only on the essential image of $G \circ F$. Given $\mathcal{L}$ in that image, extract coarse exterior parameters (M, J, Q) from the coded asymptotic content at the fixed resolution and reconstruct the corresponding Kerr–Newman exterior, $H(\mathcal{L}) := \mathrm{DOC}(\mathrm{KN}(M, J, Q))$, under standard uniqueness hypotheses.

The technical statements concerning each functor—non-dependence on vacuum choice for F, recursive equivalence for G, the canonical/non-forced character of H, and the explicit treatment of interior encoding—are recorded in Appendix G (Remarks G.1, G.2, G.5, G.4).

Triadic retraction

Proposition 2.16 (Triadic retraction on exterior data). *On the physically realized subcategory (i.e., restricting $\mathbf{Cat}_G$ to Kerr–Newman exteriors under the standard hypotheses, and $\mathbf{Cat}_L$ to the essential image of $G \circ F$), the composition $H \circ G \circ F$ recovers the exterior geometry up to the decoding resolution:*

$$H\big(G(F(\mathrm{DOC}, g))\big) \;\cong\; (\mathrm{DOC}, g),$$

where $\cong$ denotes isometry/diffeomorphism equivalence at the fixed coarse resolution. Similarly, $G \circ F \circ H$ returns a system bi-interpretable to the original $\mathcal{L}$ within that subcategory.

The proof and the precise notion of equivalence used here are given in Appendix G. The proposition records the precise sense in which the triadic diagram "commutes": a controlled equivalence on a restricted class of objects, up to the explicitly stated equivalences (isometry, gauge, recursive equivalence of codings).

Preservation of the incompleteness pattern

Theorem 2.17 (Preservation of the incompleteness pattern). *Within the above framework, and under the assumptions specified in Sections 3–4 (domain restriction, locality, admissible state class, and the physical diagonal construction), the following implications hold:*

(a) *A geometric causal barrier inducing exterior domain restriction implies the existence of nontrivial observational indistinguishability classes relative to* Obs_O.

(b) *Nontrivial observational indistinguishability classes imply the existence of domain-restricted non-derivability statements in Φ_{ext} under the arithmetic interpretation.*

Conversely, under standard locality assumptions and the definition of Obs_O *as the algebra of domain-supported observables, domain-restricted descriptive non-uniqueness entails the existence of distinct global realizations agreeing on all domain observables, which is the operational signature of causal-geometric inaccessibility.*

Corollary 2.18 (Unified incompleteness principle; conditional form). *A domain-restricted physical description Φ_{ext} is "complete" (in the ontological descriptive sense of this paper) only if: (i) the induced observational equivalence relation on physically realizable states is trivial (no nontrivial indistinguishability classes), and (ii) the corresponding encoded formal system has no Gödel-type undecidable sentences relative to the induced provability predicate. Under the triadic correspondence, failure of either condition manifests as the causal-geometric incompleteness analyzed in the main theorems.*

Extension to dynamical black holes. The triadic correspondence is formulated for stationary, asymptotically flat black hole exteriors, where uniqueness theorems allow parameterization by (M, J, Q). For dynamical black holes formed by gravitational collapse, stationarity need not hold during formation. Nevertheless, the incompleteness mechanism does not require the full stationary classification: the essential ingredients are (a) a causal barrier inducing domain restriction for exterior observers, (b) sufficient interior encoding capacity, and (c) semiclassical self-consistency (small backreaction at the relevant scales). The detailed treatment of the dynamical case (formation phase versus stationary tail) follows from the generalizations of §6.

Essential uniqueness of the correspondence. The triadic correspondence is "essentially unique" in the precise sense that the incompleteness mechanism and theorems proved later are invariant under (a) isometries/diffeomorphisms of the exterior geometry, (b) gauge/local-covariant equivalences of the physical description, and (c) recursive equivalences of effective codings. The paper does not require, and does not claim, a global equivalence of categories in a literal identity sense across all possible objects; the precise scope of the categorical claims is specified in Appendix G.

2.9 Reduction to Standard Quantum Theory in Barrier-Free Domains

A potential concern is that the incompleteness mechanism might be an artifact of the formalization rather than a consequence of causal structure. The correct diagnostic is the presence (or absence) of absolute causal barriers relative to the observational domain.

Barrier-free spacetimes/domains. If the observational domain D coincides (for the relevant purposes) with the whole region supporting the observables under consideration—so that there is no causal complement carrying independent degrees of freedom that are forever inaccessible from D—then the specific causal-geometric underdetermination mechanism analyzed here does not arise. In particular, in Minkowski spacetime with D taken to be all of spacetime (or all of a Cauchy development), the exterior/interior split used in the black-hole case is absent.

Remark 2.19. This does not assert that "quantum mechanics is complete" in an absolute philosophical sense. It asserts only that the specific incompleteness mechanism proved in this paper requires causal-domain restriction induced by causal barriers. Without such barriers, the key step—existence of distinct global states that agree on all observables in D—need not hold in the same structural way.

2.9.1 Causal-Geometric Incompleteness

Definition 2.20 (Causal-geometric incompleteness; domain-relative)**.** Let Φ_D be a physical theory formalized and applied relative to a fixed observational domain D (equivalently, relative to the algebra of observables Obs_D supported in D). We say that Φ_D exhibits *causal-geometric incompleteness* if there exist physically admissible states s_1, s_2 and a region R such that:

(i) *Causal separation:* $R \cap J^-(D) = \varnothing$ (so R is causally inaccessible from D).

(ii) *Observational equivalence on D:* $\langle A\rangle_{s_1} = \langle A\rangle_{s_2}$ for all $A \in \mathrm{Obs}_D$.

(iii) *Physical distinction:* $s_1 \neq s_2$ as global states (e.g., they disagree on some observable localized in or supported by R).

(iv) *Domain-restricted underdetermination:* Within the domain-restricted theory Φ_D, no derivation grounded solely in Obs_D can yield a statement that uniquely singles out s_1 over s_2 (or conversely) among physically admissible global states.

(v) *Causal origin (diagnostic condition):* If one enlarges the domain to D' with $R \subset J^-(D')$ (so the barrier is removed with respect to D'), then the observational equivalence in (ii) can be broken by some observable in $\mathrm{Obs}_{D'}$, i.e., the underdetermination is removed by eliminating the causal restriction.

Remark 2.21. This definition does not require pointwise classical energy conditions (WEC/DEC), which are not generally stable in QFT. Physical admissibility is instead taken to mean: membership in the chosen admissible state class (e.g., Hadamard where required), and self-consistency of the semiclassical regime where invoked (small backreaction in the explicit examples).

2.10 From Causal Barriers to Domain-Restricted Theoretical Limitations

The bridge between geometry and theory is the following: causal barriers induce domain restriction; domain restriction induces observational equivalence classes; observational equivalence classes imply non-uniqueness of domain-restricted description.

Definition 2.22 (Exterior/domain-restricted theory)**.** Let Φ be a theory with an associated observable algebra Obs and let D be an observational domain. Define Φ_D to be the restriction of Φ to the language and inferences whose empirical content is determined solely by observables in Obs_D (observables supported in D). Formally, Φ_D is the subtheory obtained by restricting the non-logical symbols for observables to those in Obs_D and restricting the admissibility/semantics accordingly.

Definition 2.23 (Domain-relative description predicate)**.** $\mathrm{Desc}_D(\Phi, s, n)$ denotes: "n is a complete description code (at the fixed resolution level) for the restriction-relevant content of state s relative to domain D," where "complete" means: n uniquely determines the equivalence class of s under observational equivalence on D within the physically admissible class.

Theorem 2.24 (Bridge theorem; existence form for Schwarzschild exteriors)**.** *Let (M, g) be a Schwarzschild spacetime and let D_O be the exterior observational domain associated with $\mathscr{I}^+$ (equivalently, the DOC exterior to the horizon). Let Obs_{D_O} be the algebra of observables supported in D_O for a fixed locally covariant QFT on (M, g), and assume the split property [47](or an equivalent technical hypothesis ensuring well-posed restriction/extension behavior for separated regions). Then there exist physically admissible global states s_1, s_2 such that:*

(i) $\langle A\rangle_{s_1} = \langle A\rangle_{s_2}$ *for all* $A \in \mathrm{Obs}_{D_O}$,

(ii) $s_1 \neq s_2$ *(they differ on interior-supported observables),*

and consequently Φ_{D_O} does not determine a unique global state up to physical equivalence.

Moreover, s_1 and s_2 can be chosen so that their exterior stress-energy expectation values agree (hence they induce the same exterior semiclassical backreaction at the chosen resolution), while their interior content differs.

Proof (outline). Step 1 (Restriction). Let $\mathfrak{A}_{\mathrm{ext}} := \mathrm{Obs}_{D_O}$ be the exterior observable algebra. Restrict any global state s to $\mathfrak{A}_{\mathrm{ext}}$ to obtain $s|_{\mathfrak{A}_{\mathrm{ext}}}$.

Step 2 (Non-uniqueness of extensions). Under standard assumptions (in particular, the split property for suitably separated exterior/interior subregions), a fixed exterior state ω_{ext} on $\mathfrak{A}_{\mathrm{ext}}$ admits multiple distinct global extensions to the full algebra $\mathfrak{A}_{\mathrm{total}}$. Concretely, one may hold ω_{ext} fixed on $\mathfrak{A}_{\mathrm{ext}}$ while varying the state on an interior commuting subalgebra (or on a type-I factor interpolating between them), producing distinct global states ω_1, ω_2 with identical restriction $\omega_1|_{\mathfrak{A}_{\mathrm{ext}}} = \omega_2|_{\mathfrak{A}_{\mathrm{ext}}} = \omega_{\mathrm{ext}}$.

Step 3 (Interior distinction). Choose two distinct interior state choices so that the resulting global extensions differ on some interior-localized observable. This yields $\omega_1 \neq \omega_2$ while preserving equality on $\mathfrak{A}_{\mathrm{ext}}$.

Step 4 (Semiclassical admissibility and backreaction control). Choose the interior variations to be localized and of sufficiently small energy in the semiclassical sense (and choose admissible states such as Hadamard states when required), so that the induced metric perturbation is parametrically small for macroscopic black holes. This ensures that the construction remains within the semiclassical regime used in the paper.

Step 5 (Domain-restricted underivability).

We argue that Φ_{D_O} cannot derive any sentence whose truth value differs between ω_1 and ω_2. The argument proceeds in two stages.

(5a) Atomic agreement. For every observable $A \in \mathfrak{A}_{\mathrm{ext}}$ and every coded rational q, the atomic statement $\mathrm{Eval}(s, A, q)$ has the same truth value in ω_1 and ω_2, because $\omega_1(A) = \omega_2(A)$ by Steps 1–3.

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(5b) Closure under logical operations. Let $\mathcal{T} \subseteq \mathrm{Sent}(\Phi_{D_O})$ denote the set of Φ_{D_O}-sentences that have identical truth values in ω_1 and ω_2 when interpreted in the standard semiclassical models built on the respective global states. By (5a), $\mathcal{T}$ contains all atomic sentences of the form $\mathrm{Eval}(s, A, q)$ with $A \in \mathfrak{A}_{\mathrm{ext}}$. Since the language of Φ_{D_O} is, by Definition 2.22, generated from observable symbols restricted to Obs_{D_O} together with the arithmetic vocabulary (whose interpretation does not depend on the choice of global extension), and since truth-value agreement is preserved under Boolean connectives and under quantification over the (shared) sorts of events, regions, and natural numbers, $\mathcal{T}$ is closed under all logical operations of the language. Hence $\mathcal{T}$ contains every well-formed sentence of Φ_{D_O}, and in particular every Φ_{D_O}-derivable sentence.

(5c) Conclusion. Suppose, for contradiction, that $\Phi_{D_O} \vdash \mathrm{Desc}_{D_O,N}(n_1)$ for the code $n_1 := \gamma_N([\omega_1])$, asserting uniqueness of ω_1 from exterior data at level N. Then by (5b), this sentence holds in both ω_1 and ω_2, contradicting Step 3 ($\omega_1 \neq \omega_2$ globally while sharing the same exterior restriction). Therefore Φ_{D_O} cannot derive a domain-relative description code $\mathrm{Desc}_{D_O,N}(\Phi, s, n)$ that uniquely singles out ω_1 over ω_2 among admissible global states consistent with the same exterior data. □

Remark 2.25 (What is proved here). The Bridge Theorem is an existence statement: in at least one standard semiclassical setting with a horizon-induced domain restriction, exterior observables underdetermine global states. This is the precise input needed later for the Gödel-type diagonal step. The theorem does not claim that every causal barrier in every theory yields the same underdetermination without further hypotheses.

2.10.1 Converse Direction

Lemma 2.26 (Converse; definitional)**.** *If Φ_D is causally-geometrically incomplete in the sense of Definition 2.20, then there exist distinct admissible states $s_1 \neq s_2$ that are observationally equivalent on D:*

$$\langle A\rangle_{s_1} = \langle A\rangle_{s_2} \quad \textit{for all } A \in \mathrm{Obs}_D\,.$$

Proof. Definition 2.20(ii) is precisely observational equivalence; Definition 2.20(iii) provides distinctness. □

Corollary 2.27 (Equivalence at the correct level of abstraction)**.** *Within the adopted framework, causal-geometric incompleteness is equivalent to the existence of nontrivial observational equivalence classes induced by causal separation. The substantive physical content is the existence of such nontrivial classes in semiclassical black hole spacetimes (Theorem 2.12).*

2.10.2 Holography (Scope-Limited Response)

A natural objection is that, in a complete quantum-gravity description with holographic duality, bulk information might be encoded in boundary degrees of freedom, potentially restoring global uniqueness. The present paper's claims are explicitly restricted to the semiclassical regime and to the domain-restricted semiclassical observable content. Even if a full quantum-gravity theory provides a boundary-to-bulk reconstruction in an appropriate setting, that does not imply that the semiclassical, domain-restricted theory Φ_D uniquely determines interior degrees of freedom from exterior semiclassical observables.

In particular: (i) the argument of this paper is formulated for asymptotically flat, semiclassical black hole spacetimes and uses the algebra of semiclassical observables accessible to an exterior domain; (ii) exact holographic dualities are established most cleanly in asymptotically AdS settings with a well-defined conformal boundary theory; (iii) the conclusion here is a limitation of the semiclassical, domain-restricted description, not a no-go theorem for quantum gravity.

Accordingly, holography—if valid in the relevant regime—should be viewed as a candidate mechanism by which a more fundamental theory may evade or reinterpret the semiclassical incompleteness, rather than as a direct refutation of the domain-restricted semiclassical result.

2.10.3 The Logical Bridge: From Indistinguishability to Undecidability

The Bridge Theorem establishes domain-restricted underdetermination: there exist $s_1 \neq s_2$ that agree on all observables in Obs_D. This yields a family of mutually consistent "completions" of Φ_D: one may consistently add a description axiom selecting s_1, or one selecting s_2, without changing any domain-observable consequences.

To obtain a specifically Gödelian pattern (as opposed to generic underdetermination), one adds the arithmetic interpretation and effective coding machinery (Sections 2.4–2.6) and then implements a fixed-point/diagonal construction (Section 4). Informally:

(1) Indistinguishability ⇒ non-unique describability. If s_1 and s_2 agree on Obs_D, then no domain-restricted derivation can produce a description code that uniquely distinguishes them using only Obs_D-content.

(2) Non-unique describability ⇒ incomplete theory. Φ_D admits multiple consistent extensions corresponding to different interior completions consistent with the same exterior theory.

(3) Gödelian strengthening via diagonalization. Using the arithmetic interpretation (Q) and the coding map γ_N, one can construct a self-referential ("fixed-point") description statement whose domain-restricted derivability is itself encoded in the interior completion. This yields the Gödel-like phenomenon: a statement (about domain-restricted describability) that is true in the intended model yet not derivable in Φ_D.

The precise diagonal construction and the exact formal statement appear in Section 4, where the required predicates (Prov, Desc_D, and the coding interface) are fixed and the fixed-point lemma is applied.

3 Black Hole Causal Structure

This section establishes that event horizons constitute absolute causal barriers (§3.1), distinguishes ontological from epistemological inaccessibility (§3.2), and formulates physically defensible admissibility and backreaction conditions for the interior encodings used later (§3.6–3.8). We also quantify the available encoding capacity from the Bekenstein–Hawking bound and delineate the semiclassical validity regime inside macroscopic black holes (§3.4).

3.1 Event Horizons as Absolute Barriers

Definition 3.1 (Event horizon)**.** Let (M, g) be an asymptotically flat spacetime with future null infinity $\mathscr{I}^+$. The (future) event horizon is

$$\mathcal{H} := \partial J^-(\mathscr{I}^+)\,.$$

The black hole region is $\mathcal{B} := M \setminus J^-(\mathscr{I}^+)$. Points in $\mathcal{B}$ cannot send causal signals to $\mathscr{I}^+$.

In Schwarzschild spacetime, with $r_\mathrm{s} = 2GM/c^2$, the region $r < r_\mathrm{s}$ lies in $\mathcal{B}$; thus it is causally disconnected from $\mathscr{I}^+$ in the outward direction.

Theorem 3.2 (One-way causal barrier in Schwarzschild)**.** *In Schwarzschild spacetime with $M > 0$, no future-directed causal curve from the interior region $R_\mathrm{int} := \{r < r_\mathrm{s}\}$ can reach the exterior region $\{r > r_\mathrm{s}\}$. Equivalently, for every p with $r(p) < r_\mathrm{s}$,*

$$J^+(p) \cap \{r > r_\mathrm{s}\} = \varnothing\,,$$

hence in particular $J^+(p) \cap \mathscr{I}^+ = \varnothing$.

Proof. We use ingoing Eddington–Finkelstein coordinates (v, r, θ, φ), in which the metric reads

$$ds^2 = -\Big(1 - \frac{r_\mathrm{s}}{r}\Big)c^2\,dv^2 + 2c\,dv\,dr + r^2\,d\Omega^2\,. \quad (9)$$

Consider radial outgoing null curves ($d\Omega = 0$, $ds^2 = 0$). Then

$$0 = -\Big(1 - \frac{r_\mathrm{s}}{r}\Big)c^2\,dv^2 + 2c\,dv\,dr$$

implies, for $dv > 0$ (future direction),

$$\frac{dr}{dv} = \frac{c}{2}\Big(1 - \frac{r_\mathrm{s}}{r}\Big)\,.$$

For $r < r_\mathrm{s}$ one has $(1 - r_\mathrm{s}/r) < 0$, hence $dr/dv < 0$. Therefore even the outward-directed null rays decrease r in the interior and cannot cross to $r > r_\mathrm{s}$. Since any future-directed timelike curve must lie inside the future null cone, it likewise cannot increase r past r_s once inside. □

Directional asymmetry. The barrier is one-way: future-directed infalling causal curves can cross from $r > r_\mathrm{s}$ to $r < r_\mathrm{s}$, but future-directed outgoing causal curves cannot cross from $r < r_\mathrm{s}$ to $r > r_\mathrm{s}$. This asymmetry is the geometric origin of domain-relative describability.

Coordinate independence. The conclusion is invariant: it follows from the global definition $\mathcal{H} = \partial J^-(\mathscr{I}^+)$ and the characterization of the black hole region $\mathcal{B} = M \setminus J^-(\mathscr{I}^+)$. The coordinate argument above provides an explicit check that, in Schwarzschild, the hypersurface $r = r_s$ coincides with that boundary and has the stated one-way causal property.

3.2 Ontological versus Epistemological Inaccessibility

Epistemological limitation: a limitation on what can be known or measured (due to technology, noise, or fundamental uncertainty relations), while the relevant degrees of freedom remain in principle within the observer's causal domain.

Ontological inaccessibility (domain restriction): a structural exclusion due to spacetime causal geometry: events/fields in a region R satisfying $R \cap J^-(D) = \varnothing$ are causally inaccessible from an observational domain D, independently of technology, computation, or measurement sophistication.

Black hole interiors are of the ontological (causal-geometric) type relative to exterior domains: the inability of exterior observers to access interior-supported information is enforced by causal structure, not by instrumental limitations.

Theorem 3.3 (Geometric necessity of the horizon barrier)**.** *For macroscopic Schwarzschild black holes ($M \gg m_P$), the one-way causal barrier at the event horizon is:*

(i) *coordinate-independent (defined geometrically by $\mathcal{H} = \partial J^-(\mathscr{I}^+)$),*

(ii) *observer-independent (all exterior observers agree on the existence of $\mathcal{H}$ as a null boundary of $J^-(\mathscr{I}^+)$),*

(iii) *technology-independent (no physical signaling protocol can transmit information from $\mathcal{B}$ to $\mathscr{I}^+$ without violating the causal structure),*

(iv) *stable in the semiclassical regime: for $M \gg m_P$, the leading semiclassical corrections to the metric are suppressed by powers of $(m_P/M)^2$ (Theorem 3.9), so that the location and existence of the horizon are perturbatively unaffected.*

Proof (sketch). (i)–(ii) are immediate from the geometric definition. (iii) is a standard consequence of global causal structure: signaling from $\mathcal{B}$ to $\mathscr{I}^+$ would imply $\mathcal{B} \subseteq J^-(\mathscr{I}^+)$, contradicting the definition of $\mathcal{B}$. (iv) is a regime statement: for $M \gg m_P$, curvature at the horizon is parametrically small and semiclassical corrections are suppressed by powers of (m_P/M). □

3.3 Visualization (Penrose Diagram; Schematic)

A conformal (Penrose) diagram of Schwarzschild spacetime schematically encodes $\mathscr{I}^+$ and $\mathscr{I}^-$ (future and past null infinity), the horizon $\mathcal{H}$ as a null hypersurface, the black hole region $\mathcal{B}$ whose future does not intersect $\mathscr{I}^+$, and the spacelike singularity at $r = 0$ as the future endpoint of interior-directed timelike curves.

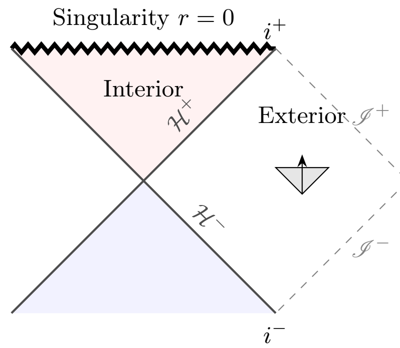

Figure 2: Penrose diagram of a Schwarzschild black hole. The diagram is schematic; only the global causal relations between $\mathscr{I}^\pm$, i^0, $i^\pm$, $\mathcal{H}^\pm$, and the singularity at $r = 0$ are represented faithfully. The conformal compactification follows the standard Schwarzschild presentation [4, Ch. 5].

3.4 Bekenstein–Hawking Entropy Bound and Encoding Capacity

Theorem 3.4 (Bekenstein–Hawking entropy; Schwarzschild)**.** *A Schwarzschild black hole of mass M has horizon area $A = 4\pi r_s^2$ with $r_s = 2GM/c^2$, and entropy*

$$S_{BH} = \frac{k_B\, c^3\, A}{4\, G\, \hbar} = \frac{4\pi\, k_B\, G\, M^2}{\hbar\, c}. \tag{10}$$

The corresponding information capacity (in bits) is $N_{bits} = S_{BH}/(k_B \ln 2)$.

Numerical examples (Schwarzschild).

M	r_s	S_{BH}/k_B	N_{bits}
$M_\odot$	2.95×10^3 m	1.05×10^{77}	1.51×10^{77}
$10^6\, M_\odot$	2.95×10^9 m	1.05×10^{89}	1.51×10^{89}
$10^9\, M_\odot$	2.95×10^{12} m	1.05×10^{95}	1.51×10^{95}

Interpretation for the later encoding. The entropy bound provides a conservative upper limit on the number of mutually distinguishable microstates compatible with fixed exterior macroscopic data: $N_{states} \lesssim \exp(S_{BH}/k_B)$. Even for stellar-mass black holes this number is astronomically large, so any finite Gödel-type coding used in the diagonal construction fits well within the available capacity.

3.4.1 Semiclassical validity region inside macroscopic black holes

Semiclassical breakdown criteria. The semiclassical approximation is expected to fail when curvature reaches the Planck scale or when backreaction becomes order unity. A convenient curvature diagnostic is the Kretschmann scalar for Schwarzschild,

$$K(r) = R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} = \frac{48\, G^2 M^2}{c^4\, r^6} = \frac{12\, r_{\rm s}^2}{r^6}\,. \tag{11}$$

At the horizon, $K(r_{\rm s}) = 12/r_{\rm s}^4$, which is extremely small for macroscopic $r_{\rm s}$.

Planck-curvature radius. Define $r_{\rm QG}$ by $K(r_{\rm QG}) \approx \ell_{\rm P}^{-4}$. Solving yields

$$r_{\rm QG} \approx 12^{1/6}\, (r_{\rm s}\, \ell_{\rm P}^2)^{1/3}\,. \tag{12}$$

For $M = M_\odot$ ($r_{\rm s} \approx 2.95\times 10^3$ m), $r_{\rm QG} \approx 1.39\times 10^{-22}$ m. The Planck-curvature core occupies a negligible fraction of the interior volume:

$$(r_{\rm QG}/r_{\rm s})^3 \approx 1.0 \times 10^{-76} \qquad (M = M_\odot)\,.$$

We restrict the support of the encoding degrees of freedom to $r_{\rm QG} < r < r_{\rm s} - \delta r$ with $\delta r \gg \ell_{\rm P}$, leaving essentially the full interior available in macroscopic cases.

Thermal noise scale (Hawking temperature). The Hawking temperature is

$$T_H = \frac{\hbar\, c^3}{8\pi\, G\, M\, k_{\rm B}}\,, \tag{13}$$

with associated frequency scale $\omega_H = k_{\rm B} T_H/\hbar = c^3/(8\pi G M)$. For $M = M_\odot$, $\omega_H \approx 8.1 \times 10^3\,{\rm s}^{-1}$. If the encoding uses modes of characteristic frequency $\omega \gg \omega_H$, then thermal occupation at ω is exponentially suppressed and does not contaminate the encoding.

3.5 Physical Realizability Timeline (Heuristic)

This subsection is motivational and not used as a premise in the proofs. The goal is to indicate that interior encoding states are not "eternally fine-tuned" artifacts.

For a stellar-mass collapse, the characteristic light-crossing time at the horizon scale is $t_s \sim r_{\rm s}/c \sim 10^{-5}$ s (for $M = M_\odot$). Formation and relaxation processes occur on multiples of t_s. A commonly used estimate for the scrambling time is

$$t_{\rm scr} \sim \frac{r_{\rm s}}{c}\, \ln(S_{\rm BH}/k_{\rm B})\,,$$

which for $M = M_\odot$ gives $t_{\rm scr}$ of order 10^{-3}–10^{-2} s. After formation, exterior observers have access only to the exterior domain; interior-supported information is causally excluded from $\mathscr{I}^+$.

3.6 Physically Admissible States and Exterior Indistinguishability Classes

In QFT on curved spacetime, pointwise classical energy conditions (WEC/DEC) are not generally stable under renormalization and can be violated by physically reasonable quantum states. For our purposes, the correct admissibility conditions are those standard in the semi-classical/algebraic setting: regularity of the state, well-defined renormalized stress-energy, and parametrically small backreaction in the regime used.

Definition 3.5 (Admissible semiclassical state; operational form)**.** A state s of the field algebra on a black hole background is *admissible* for this paper if it belongs to one of the following two classes:

Class I (Pure Hadamard states). s is a pure quasi-free state of Hadamard type, satisfying:

(I.i) the two-point function $\langle \phi(x)\phi(y)\rangle_s$ has the standard Hadamard singularity structure, ensuring a well-defined renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle_s^{\rm ren}$,

(I.ii) the field equations are satisfied in the distributional/algebraic sense,

(I.iii) the induced semiclassical backreaction is perturbatively small ($\delta g/g \ll 1$; see §3.8),

(I.iv) s is compatible with the chosen exterior macroscopic data (the values of M, J, Q fixed at the resolution level N of the coding scheme; cf. §2.4).

Class II (Mixed admissible states). s is a normal state on the global observable algebra that is locally quasi-equivalent to a Hadamard state on every relatively compact subregion (in particular, $\langle T_{\mu\nu}\rangle_s^{\rm ren}$ is well-defined), and satisfies (I.ii)–(I.iv).

The admissibility class $\mathcal{S}_{\rm adm}$ is the union of Classes I and II. Class II is closed under convex mixtures; Class I is not. Both classes embed naturally into the GNS representation of any reference Hadamard state, and the local quasi-equivalence condition guarantees that all admissible states share the same ultraviolet structure required for renormalization [6, 40].

Theorem 3.6 (Nontrivial exterior indistinguishability classes; continuous family)**.** *Let D_O be an exterior observational domain and let Obs_{D_O} denote the algebra of observables supported in D_O. Suppose there exist two distinct admissible global states $s_1 \neq s_2$ whose restrictions to Obs_{D_O} coincide: $s_1|_{\mathrm{Obs}_{D_O}} = s_2|_{\mathrm{Obs}_{D_O}}$. Then the observational indistinguishability class over that exterior restriction contains a* continuously parameterized family *of physically distinct admissible states:*

$$\{\, s_t \,:\, t \in T \,\} \subseteq \{\, s : s\big|_{\mathrm{Obs}_{D_O}} = s_1\big|_{\mathrm{Obs}_{D_O}} \},\tag{14}$$

where T is either the open interval $(0,1)$ (Class II convex mixtures) or an open disk in $\mathbb{C}$ (Class I coherent excitations), and the map $t \mapsto s_t$ is injective. Equivalently, the indistinguishability class has cardinality at least $2^{\aleph_0}$.

Proof. We construct $2^{\aleph_0}$ distinct admissible states sharing the same exterior restriction by two complementary methods, depending on the regularity class invoked.

Method 1 (Convex mixtures within Class II). For each $\lambda \in (0,1)$, define $s_\lambda := \lambda\, s_1 + (1-\lambda)\, s_2$. By Definition 3.5, s_λ is a Class II admissible mixed state: it is a normal state on the global algebra, locally quasi-equivalent to a Hadamard reference state on every relatively compact region (since s_1, s_2 both have this property and quasi-equivalence is preserved under convex mixtures of normal states [46]), and inherits properties (I.ii)–(I.iv) by linearity. Moreover, $s_\lambda|_{\mathrm{Obs}_{D_O}} = s_1|_{\mathrm{Obs}_{D_O}}$ for all λ, while distinct $\lambda \neq \lambda'$ yield distinct states by the Hahn–Banach separation property. Since $|(0,1)| = 2^{\aleph_0}$, this already establishes the conclusion when the admissibility class is taken to be the union of Classes I and II.

Method 2 (Continuous family of pure Hadamard states). Suppose one wishes to restrict attention to Class I (pure Hadamard) states. Let $f \in C_c^\infty(R_{\mathrm{int}})$ be a smooth test function compactly supported in the interior region R_{int}, and let $\hat{a}^\dagger(f), \hat{a}(f)$ denote smeared creation/annihilation operators in the GNS representation of s_1. For each $\alpha \in \mathbb{C}$, define the coherent excitation

$$s_\alpha := s_1 \circ \mathrm{Ad}\big(\mathcal{D}(\alpha f)\big), \qquad \mathcal{D}(\alpha f) := \exp\big(\alpha\, \hat{a}^\dagger(f) - \bar{\alpha}\, \hat{a}(f)\big).$$

Each s_α is a pure Hadamard state: coherent displacements by $\mathcal{D}(\alpha f)$ shift the two-point function by a classically smooth term (the coherent amplitude $\langle\phi\rangle_\alpha$) while leaving the singular part of the Hadamard expansion invariant [40, 37]. Consequently the wavefront set of the two-point function is unchanged and s_α remains Hadamard (see also [6], Ch. 4).

Moreover, $s_\alpha|_{\mathrm{Obs}_{D_O}} = s_1|_{\mathrm{Obs}_{D_O}}$. This follows not from microcausality alone—which would require R_{int} and R_{ext} to be causally disjoint, a condition that fails in Schwarzschild since exterior events can send signals into R_{int}—but from the split property of Lemma B.1: under the buffered separation with $\delta \gg \ell_{\mathrm{P}}$, the displacement $\mathcal{D}(\alpha f)$ supported in R^δ_{int} acts on the buffered exterior algebra as the identity, up to corrections suppressed by the collar scale δ. Hence for any $A \in \mathfrak{A}_{\mathrm{ext}}$:

$$s_\alpha(A) = \big(s_1 \circ \mathrm{Ad}(\mathcal{D}(\alpha f))\big)(A) = s_1\big(\mathcal{D}(\alpha f)^* A\, \mathcal{D}(\alpha f)\big) = s_1(A)\,, \tag{15}$$

where the equality holds up to the collar-scale corrections quantified in Lemma B.1. For sufficiently small $|\alpha|$, the backreaction condition (I.iii) is satisfied (Theorem 3.9). Distinct $\alpha \neq \alpha'$ yield distinct states because the coherent expectation values $\langle \hat{a}(f)\rangle_{s_\alpha} = \alpha\, \|f\|_2^2$ differ. Since the disk $\{|\alpha| < \alpha_{\max}\} \subset \mathbb{C}$ has cardinality $2^{\aleph_0}$, the conclusion follows within Class I alone.

In either case, the indistinguishability class has cardinality at least $2^{\aleph_0}$.

□

Remark 3.7. The Bridge Theorem (Theorem 2.12) provides sufficient conditions (e.g. split property, existence of distinct extensions) that guarantee the hypothesis "there exist $s_1 \neq s_2$ with identical exterior restriction" in standard semiclassical settings. The present theorem isolates the purely set-theoretic conclusion once that hypothesis is established.

3.7 Stress-Energy Scale for Interior Encoding

To ensure that the encoding states used later remain semiclassical, we impose an explicit stress-energy scale control that implies small ADM mass shift and hence small geometric perturbation.

Lemma 3.8 (Energy scale of an interior encoding)**.** *Consider an interior excitation built from n_q quanta in modes of characteristic frequency ω, so that the total excitation energy is bounded by $E_{\mathrm{int}} \lesssim n_q\, \hbar\omega$. A conservative semiclassical requirement is $E_{\mathrm{int}} \ll Mc^2$, which guarantees that the induced ADM mass shift $\delta M = E_{\mathrm{int}}/c^2$ is tiny compared to M. In the canonical estimate $\omega \sim c/r_{\mathrm{s}}$, one has $E_{\mathrm{int}} \lesssim n_q\, \hbar c/r_{\mathrm{s}}$.*

3.8 Backreaction Analysis: Preservation of the Semiclassical Geometry

The backreaction problem. The semiclassical Einstein equation couples geometry to the renormalized stress-energy:

$$G_{\mu\nu}[g] = \frac{8\pi G}{c^4}\, \langle T_{\mu\nu}\rangle\,.$$

If interior encoding significantly increases $\langle T_{\mu\nu}\rangle$ (in a way visible at infinity as an ADM mass shift), it could move the horizon and invalidate the domain restriction. We therefore quantify conditions under which the perturbation is negligible.

Perturbative framework. Write $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, where $g^{(0)}_{\mu\nu}$ is the Schwarzschild metric and $\|h_{\mu\nu}\| \ll 1$ in an appropriate norm on the exterior domain.

Theorem 3.9 (Energy-based backreaction bound; parametric form)**.** *Let a Schwarzschild black hole of mass $M \gg m_{\mathrm{P}}$ support an admissible interior excitation whose ADM mass shift is $\delta M = E_{\mathrm{int}}/c^2$. Then, at the level of parametric (order-of-magnitude) control in linearized gravity,*

$$\frac{\delta g}{g} \sim \frac{\delta r_{\mathrm{s}}}{r_{\mathrm{s}}} = \frac{\delta M}{M} = \frac{E_{\mathrm{int}}}{Mc^2}\,. \tag{16}$$

In particular, if $E_{\mathrm{int}} \lesssim n_q\, \hbar\omega$ and $\omega \sim c/r_{\mathrm{s}}$, then

$$\frac{\delta g}{g} \lesssim \frac{n_q\, \hbar c}{r_{\mathrm{s}}\, Mc^2} = \frac{n_q\, \hbar c}{2GM^2} = \frac{n_q}{2}\Big(\frac{m_{\mathrm{P}}}{M}\Big)^2. \tag{17}$$

Numerical estimate ($M = M_\odot$). For $M = M_\odot$, $(m_{\mathrm{P}}/M_\odot)^2 \approx 1.2\times 10^{-76}$, hence $\delta g/g \lesssim O(10^{-76}) \times n_q$. For $n_q = 10^6$, $\delta g/g \lesssim O(10^{-70}) \ll 1$. The semiclassical geometry and the location of the horizon are unaffected at any physically relevant scale. □

Corollary 3.10 (Horizon stability)**.** *Since $r_{\mathrm{s}} \propto M$, one has $\delta r_{\mathrm{s}} = r_{\mathrm{s}}\,(\delta M/M) = r_{\mathrm{s}}\, E_{\mathrm{int}}/(Mc^2)$. For $M = M_\odot$ and $n_q = 10^6$ with $\omega \sim c/r_{\mathrm{s}}$,*

$$\delta r_{\mathrm{s}} \lesssim r_{\mathrm{s}} \times O(10^{-70}) \approx O(10^{-67})\,\mathrm{m}\,,$$

which is far below ℓ_{P}. The causal barrier defined by the horizon remains unchanged at semiclassical resolution.

Corollary 3.11 (Semiclassical validity for the encoding regime)**.** *For macroscopic black holes ($M \gg m_{\mathrm{P}}$), any interior encoding satisfying $E_{\mathrm{int}} \ll Mc^2$ (equivalently $\delta g/g \ll 1$) lies within the semiclassical regime on the exterior domain and does not spoil the causal-domain restriction used in the incompleteness construction.*

3.9 Three Levels of Physical Realizability (Interpretive)

It is useful to distinguish three nested notions of physical realizability for the states used in this paper. Level 1 coincides with $\mathcal{S}_{\rm adm}$ of Definition 3.5: Hadamard/regularity, well-defined renormalized $\langle T_{\mu\nu}\rangle$, and controlled backreaction. Level 2 strengthens this to states obtainable via realistic collapse/accretion histories. Level 3 further restricts to states realizable in laboratory analog systems. The theorems of this paper require only Level 1; Levels 2–3 are not used in any proof.

3.10 Universality of the Causal Barrier for Asymptotically Flat Black Holes

For stationary, asymptotically flat black holes (Schwarzschild, Kerr, Reissner–Nordström, Kerr–Newman), the event horizon is defined geometrically by $\mathcal{H} = \partial J^-(\mathscr{I}^+)$, and the associated black hole region $\mathcal{B} = M \setminus J^-(\mathscr{I}^+)$ is causally disconnected from $\mathscr{I}^+$ in the outward direction.

Theorem 3.12 (Universality of the causal barrier)**.** *In any asymptotically flat black hole spacetime with event horizon $\mathcal{H} := \partial J^-(\mathscr{I}^+)$, every point p in the black hole region $\mathcal{B} := M \setminus J^-(\mathscr{I}^+)$ satisfies*

$$J^+(p) \cap \mathscr{I}^+ = \varnothing\,.$$

In particular, interior-supported information cannot be transmitted to future null infinity.

Proof. By definition $\mathcal{B} = M \setminus J^-(\mathscr{I}^+)$, so for any $p \in \mathcal{B}$ we have $p \notin J^-(\mathscr{I}^+)$, equivalently $J^+(p) \cap \mathscr{I}^+ = \varnothing$. The result is therefore a tautological consequence of the geometric definition of the black hole region; it does not depend on stationarity, asymptotic flatness beyond what is required to define $\mathscr{I}^+$, or any uniqueness assumption. □

Remark 3.13. This theorem uses only the geometric definition of event horizon and black hole region; it does not require no-hair classification. No-hair/uniqueness enters elsewhere only when one wants a clean parameterization of stationary exteriors by (M, J, Q).

The geometric structure established in this section—absolute one-way causal barriers, ontological domain restriction, and Hadamard-admissible interior states with negligible backreaction—is the antecedent of the Master Incompleteness Theorem (§4.12). The implications for theories that modify any of these features are discussed in §9.4.

4 Construction of an Undecidable State s^*

This section establishes the incompleteness claim by (i) formulating a domain-restricted notion of "complete describability" grounded in exterior observables, (ii) implementing a Gödel-style fixed-point (diagonal) construction inside the arithmetic interpreted by $\Phi_{\rm semi}$, and (iii) using the causal barrier to show that the resulting statement is undecidable relative to the exterior-restricted theory. Throughout, we avoid assuming an exact tensor factorization $\mathcal{H}_{\rm total} = \mathcal{H}_{\rm ext} \otimes \mathcal{H}_{\rm int}$ and instead work with the restriction/extension of states on local observable algebras (split property where needed; cf. §2.10 and §3.6).

4.1 Adaptation of the Diagonal Lemma to a Causal Domain

Classical diagonal lemma (Gödel 1931). For any arithmetic formula $\varphi(x)$ with one free variable, there exists a sentence G such that

$$\mathrm{PA} \vdash G \;\leftrightarrow\; \varphi(\ulcorner G \urcorner)\,,$$

where $\ulcorner G \urcorner$ is the Gödel number of G.

Exterior-restricted physical setting. Fix an exterior observational domain D_O (e.g. the DOC associated with $\mathscr{I}^+$ in Schwarzschild) and let Obs_{D_O} denote the algebra of observables supported in D_O. Let Φ_{D_O} denote the restriction of semiclassical theory to inferences whose empirical content is grounded in Obs_{D_O} (Definition 2.22).

Coding at finite resolution. Fix a finite-resolution coding γ_N of admissible physical states into $\mathbb{N}$ (as in §2.4–2.6), and let $\mathrm{Ad}_N(n)$ be the predicate "n codes an admissible state at resolution N."

Exterior indistinguishability predicate. Choose a countable separating family of smeared exterior observables $\{A_i\}_{i\in\mathbb{N}} \subset \mathrm{Obs}_{D_O}$ (standard in algebraic QFT). Define a finite-resolution exterior agreement predicate:

$$\mathrm{SameExt}_N(n,m) \equiv$$
"for all $i \le N$, the predicted expectation values of A_i in n, m agree within $\pm\varepsilon_N$"

where ε_N is the fixed discretization tolerance at level N.

Domain-relative completeness predicate. Define the domain-relative "complete describability" predicate at level N by:

$$\mathrm{Desc}_{D_O,N}(n) \;:\equiv\; \forall m\left[\,\mathrm{Ad}_N(m) \;\wedge\; \mathrm{SameExt}_N(n,m) \;\rightarrow\; m = n\,\right]. \quad (18)$$

Thus $\mathrm{Desc}_{D_O,N}(n)$ asserts: "within the admissible class at resolution N, the exterior data (up to level N) uniquely determines the coded state n."

Representability assumption. By §2.6 (arithmetic interpretation) we assume that the predicates Ad_N, $\mathrm{SameExt}_N$, $\mathrm{Desc}_{D_O,N}$ are representable in the formal system associated with $\Phi_{\rm semi}$ (and hence expressible in the language used by Φ_{D_O} via the coding interface).

This assumption is justified by the bounded character of the predicates at each fixed resolution level N: Ad_N checks membership in a finite list, $\mathrm{SameExt}_N$ compares finitely many rational-valued expectations within tolerance ε_N, and $\mathrm{Desc}_{D_O,N}$ applies a bounded universal quantifier over these comparisons. The exterior observables $\{A_i\}$ entering $\mathrm{SameExt}_N$ may be chosen as a subfamily of the coding observables $\{O_i\}$ (or, equivalently, the coding family may be enriched to contain a separating family of exterior observables); under this convention, comparison of expectation values reduces to comparison of finitely many rationals at the chosen resolution. Each predicate is therefore Δ_0 (bounded), and all Δ_0 predicates are representable in any theory interpreting Robinson Arithmetic Q (cf. [33], Ch. IV). This is the only logical input needed to apply diagonalization.

Application of the diagonal lemma to Φ_{D_O}. Let $\mathrm{Prov}_{D_O}(x)$ be the standard arithmetized provability predicate for Φ_{D_O}. Consider the arithmetic formula

$$\varphi(x) \;:\equiv\; \neg\mathrm{Prov}_{D_O}\big(\ulcorner \mathrm{Desc}_{D_O,N}(x)\urcorner\big)\,,$$

i.e., "Φ_{D_O} does not prove $\mathrm{Desc}_{D_O,N}(x)$." By the diagonal lemma (applied inside the interpreted arithmetic), there exists a sentence G^* such that

$$\Phi_{D_O} \vdash G^* \;\leftrightarrow\; \neg\mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)\,. \tag{19}$$

Let $n^* := \ulcorner G^*\urcorner$. We then select an admissible physical state s^* with code $\gamma_N(s^*) = n^*$, and refer to s^* as the *fixed-point state*.

Remark 4.1 (Why this is the right analogue). Gödel's G states its own unprovability. Here the fixed-point sentence G^* states its own unprovability in Φ_{D_O}. The physical content enters because (a) provability in Φ_{D_O} is restricted by the causal domain D_O, and (b) codes n correspond to physically admissible states via γ_N. The horizon is what makes the restriction nontrivial.

Lemma 4.2 (Finite-resolution coding separates physically distinct classes). *At fixed resolution N, if two admissible coded states $n \neq m$ both satisfy* $\mathrm{Desc}_{D_O,N}(n)$ *and* $\mathrm{Desc}_{D_O,N}(m)$*, then their exterior predictions differ at level N:* $\neg\mathrm{SameExt}_N(n,m)$.

Proof. If $\mathrm{SameExt}_N(n,m)$ held, then $\mathrm{Desc}_{D_O,N}(n)$ would imply $m = n$, contradiction. □

4.2 Construction of the Fixed-Point State s^* in Schwarzschild

Setup. Consider a Schwarzschild black hole of mass $M = M_\odot$ and an exterior domain D_O given by the DOC. Let Obs_{D_O} be the exterior observable algebra.

Step 1 (Exterior restriction rather than Hilbert factorization). We do not assume $\mathcal{H}_{\rm total} = \mathcal{H}_{\rm ext} \otimes \mathcal{H}_{\rm int}$ as an exact identity. Instead, for any global state ω (a positive normalized linear functional on the global algebra), denote its exterior restriction by $\omega_{\rm ext} := \omega|_{\mathrm{Obs}_{D_O}}$.

Step 2 (Existence of distinct global extensions with identical exterior restriction). By the Bridge Theorem (Theorem 2.12), under standard technical hypotheses (split property / well-posed restriction–extension), there exist distinct admissible global states $\omega_1 \neq \omega_2$ such that $\omega_1|_{\mathrm{Obs}_{D_O}} = \omega_2|_{\mathrm{Obs}_{D_O}}$.

Step 3 (Selecting the fixed-point code). Let n^* be the fixed-point code produced in §4.1, and select an admissible global state ω_{n^*} obtained from a fiducial reference state by an interior-supported modification (cf. Method 2 of Theorem 3.6), whose code satisfies $\gamma_N(\omega_{n^*}) = n^*$. Denote the corresponding coded physical state by s^*.

Step 4 (Physical admissibility and backreaction control). Choose ω_{n^*} within the admissible class of §3.6: regular (e.g. Hadamard where required), well-defined renormalized $\langle T_{\mu\nu}\rangle$, and with excitation energy $E_{\rm int}$ small enough that $\delta g/g \ll 1$ (Theorem 3.9 / Corollary 3.11). This guarantees that the causal barrier (horizon) persists at semiclassical resolution and that the exterior domain D_O is well-defined for the construction.

Theorem 4.3 (Existence of admissible states realizing arbitrary codes)**.** *For a macroscopic Schwarzschild black hole ($M \gg m_{\rm P}$) and any fixed resolution level N, for every code $n \in \mathbb{N}$ satisfying $n \le \exp(S_{\rm BH}/k_{\rm B})$, there exists an admissible state s_n with $\gamma_N(s_n) = n$. In particular, for the Gödel fixed point n^* of §4.1, there exists an admissible state s^* with $\gamma_N(s^*) = n^*$.*

Proof. The coding γ_N encodes the N-tuple of discretized expectation values $(Q_1(\langle O_1\rangle_s), \ldots, Q_N(\langle O_N\rangle_s))$ via prime-power factorization (Eq. (6)). For each target code n, one decodes n into a rational N-tuple $(q_1, \ldots, q_N)$. The task reduces to constructing an interior excitation whose expectation values match $(q_1, \ldots, q_N)$ within the discretization tolerance.

Step 1 (Available degrees of freedom). By the split property (Lemma B.1), the interior algebra $\mathfrak{A}_{\rm int}$ admits a type-I factor interpolation with $\mathfrak{A}_{\rm ext}$, so interior states can be varied independently of any fixed exterior restriction.

Step 2 (Reachability of target expectation values). Each gauge-invariant observable $\hat{O}_i$ restricted to the interior takes a continuum of values on the admissible state space (since the interior Fock space supports states $\sum_k c_k\,|k\rangle$ with adjustable coefficients). By varying the interior state within the admissible class $\mathcal{S}_{\rm adm}$ (Definition 3.5), which contains both pure Hadamard states (Class I) and convex mixtures of Hadamard-quasi-equivalent states (Class II), one can achieve any target tuple $(q_1, \ldots, q_N) \in \mathbb{Q}^N$ within the discretization bins. Density of the achievable set follows from the convexity of Class II together with the fact that pure Hadamard excitations span a dense subset of the regular state space (cf. Lemma B.5), provided the energy of the required excitation does not exceed Mc^2.

Step 3 (Energy bound). The code n^* is the Gödel number of a sentence of length ℓ in the language of Φ_0. Standard Gödel numbering yields $n^* \le p_\ell^{c_0}$ for a constant c_0 depending on the alphabet size. For any fixed N

and ℓ, this is a finite (though potentially large) natural number. The corresponding interior excitation requires at most $n_q \lesssim N \cdot \lceil \log_2 n^* \rceil$ quanta at frequency $\omega \sim c/r_s$, giving $E_{\rm int} \lesssim n_q \hbar c/r_s$. Since n^* is a fixed finite number (independent of M) while $Mc^2 \to \infty$ as $M/m_P \to \infty$, the backreaction condition $E_{\rm int} \ll Mc^2$ is satisfied for sufficiently macroscopic black holes.

Step 4 (Hadamard regularity). The interior excitations are constructed as finite-particle states above a reference Hadamard state (e.g., the Unruh state restricted to the interior). Finite-particle excitations of Hadamard states remain Hadamard [6, 40], so the admissibility conditions of Definition 3.5 are preserved. □

4.2.1 Physical Mechanism of Self-Reference (Structural, Not Dynamical)

Objection. The fixed-point state s^* may appear "put in by hand."

Response (scope). The claim is existential and structural, not dynamical. The construction does not require that generic collapse dynamics produce s^*. It requires only that (i) the interior admits enough admissible microstates to realize codes, and (ii) the horizon enforces a domain restriction that prevents exterior proofs from accessing interior-supported distinguishing information.

Mechanism (structural). (1) *Logical fixed point:* diagonalization produces a sentence that refers (arithmetically) to its own provability in Φ_{D_O}. (2) *Physical realization:* the code n^* is realized as an admissible interior modification at negligible backreaction. (3) *Operational closure:* the causal barrier ensures that Φ_{D_O}-derivations grounded in Obs_{D_O} cannot "see" the interior degrees of freedom that implement n^*.

This is the precise sense in which the self-reference is geometric/operational rather than syntactic-in-vacuo.

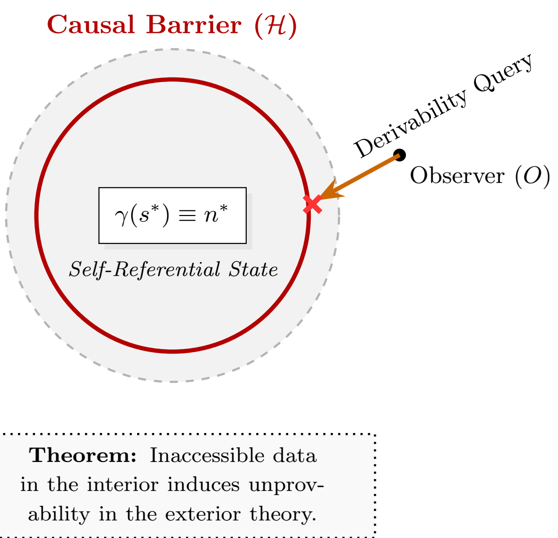

Figure 3: Schematic of the Physical Diagonal Lemma: Geometric enforcement of logical incompleteness.

4.3 Undecidability in the Exterior-Restricted Theory

We now state the undecidability claim, working under the standard Gödel hypotheses corresponding to (H3) consistency and (H4) Σ_1-soundness of the Master Theorem (§4.12).

Assumption C (Hilbert–Bernays–Löb derivability conditions). The provability predicate $\mathrm{Prov}_{D_O}(x)$ in Φ_{D_O} satisfies the standard Hilbert–Bernays–Löb (HBL) derivability conditions [33]:

(D1) If $\Phi_{D_O} \vdash \varphi$, then $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner\varphi\urcorner)$.

(D2) $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner\varphi \to \psi\urcorner) \to \big(\mathrm{Prov}_{D_O}(\ulcorner\varphi\urcorner) \to \mathrm{Prov}_{D_O}(\ulcorner\psi\urcorner)\big)$.

(D3) $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner\varphi\urcorner) \to \mathrm{Prov}_{D_O}(\ulcorner\mathrm{Prov}_{D_O}(\ulcorner\varphi\urcorner)\urcorner)$.

These conditions are automatically satisfied by any theory that interprets Robinson Arithmetic Q with a Σ_1-representable proof relation, hence are guaranteed by Hypothesis (H1) of Theorem 4.14.

Theorem 4.4 (Gödel-type undecidability for the fixed–point sentence)**.** *Let G^* be the fixed-point sentence of §4.1 with $n^* = \ulcorner G^* \urcorner$ and let s^* be an admissible state with $\gamma_N(s^*) = n^*$. Under Assumptions A–B,*

$$\Phi_{D_O} \nvdash G^* \qquad \textit{and} \qquad \Phi_{D_O} \nvdash \neg G^* . \tag{20}$$

Proof (standard Gödel pattern with HBL conditions). By construction of the diagonal fixed point (Eq. (19)), $\Phi_{D_O} \vdash G^* \leftrightarrow \neg\mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)$.

First part: $\Phi_{D_O} \nvdash G^*$. Suppose, for contradiction, that $\Phi_{D_O} \vdash G^*$. By condition (D1) of Assumption C, this entails $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)$. On the other hand, from $\Phi_{D_O} \vdash G^*$ and the biconditional, we obtain $\Phi_{D_O} \vdash \neg\mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)$. The two derivations together yield $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner) \wedge \neg\mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)$, contradicting consistency (Assumption A). Hence $\Phi_{D_O} \nvdash G^*$.

Second part: $\Phi_{D_O} \nvdash \neg G^*$. Suppose, for contradiction, that $\Phi_{D_O} \vdash \neg G^*$. By the biconditional, $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)$. The formula $\mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner) \equiv \exists p\, \mathrm{Proof}_{D_O}(p, \ulcorner G^*\urcorner)$ is a Σ_1-statement asserting the existence of a proof code. Under Σ_1-soundness (Assumption B), if $\Phi_{D_O} \vdash \mathrm{Prov}_{D_O}(\ulcorner G^*\urcorner)$, then this Σ_1-statement is true in the standard model, i.e., there exists an actual Φ_{D_O}-proof of G^*. This contradicts the first part. Therefore $\Phi_{D_O} \nvdash \neg G^*$. □

Physical reading. G^* is a statement about what Φ_{D_O} can prove given only the exterior domain. Its undecidability is a precise Gödelian limitation internal to the domain-restricted formalization.

Corollary 4.5 (Non-derivability of domain-complete exterior descriptions for s^*)**.** *In particular, Φ_{D_O} does not prove that the code n^* is exterior-complete at level N:*

$$\Phi_{D_O} \nvdash \mathrm{Desc}_{D_O,N}(n^*) \,.$$

Remark 4.6. This corollary captures the physically relevant claim: the exterior-restricted theory cannot establish uniqueness of the interior-coded state from exterior data. It does not rely on the ambiguous notion "s^* is observable"; it relies on the domain-relative uniqueness predicate $\mathrm{Desc}_{D_O,N}$.

Remark 4.7 (Levels of analysis: geometric underdetermination versus Gödelian overlay). The core incompleteness mechanism is purely geometric/operational: restricting attention to an observer's accessible algebra of observables yields a non-injective restriction map on the state space, hence multiple physically distinct interior extensions share identical exterior data. This "underdetermination-by-restriction" already implies non-uniqueness of description relative to exterior evidence.

The Gödelian terminology is an additional, structure-preserving formalization: once one fixes (i) a recursively enumerable presentation of the exterior-restricted theory Φ_{ext}, and (ii) an explicit effective encoding map γ from physically distinguishable state-data to $\mathbb{N}$, one can implement a diagonal/fixed-point pattern in the logical representation. Accordingly, the geometric indistinguishability result (state non-uniqueness under restriction) is the primary physical statement; the diagonal construction serves to exhibit a canonical self-referential representative s^* within the resulting equivalence classes, under explicit representability and encoding assumptions.

4.4 Geometric Self-Reference (Operational Form)

The "self-reference" is realized by the composite structure: (i) arithmetic coding γ_N of admissible physical states, (ii) arithmetized provability Prov_{D_O} internal to Φ_{D_O}, (iii) a causal barrier ensuring that Φ_{D_O} is genuinely domain-restricted.

Definition 4.8 (Geometric/operational self-reference). A configuration exhibits *geometric self-reference* (relative to D_O) if there exists a code n such that the corresponding fixed-point sentence G_n refers to $\mathrm{Prov}_{D_O}(\ulcorner G_n \urcorner)$, while n is physically realized by interior degrees of freedom that are causally excluded from D_O.

Theorem 4.9 (Black holes realize operational self-reference). *Schwarzschild black holes (and, more generally, spacetimes with event horizons relative to $\mathscr{I}^+$) admit operational self-reference in the above sense: there exist admissible states s^* whose codes implement a Gödel fixed point for the exterior-restricted theory.*

Proof. Diagonalization supplies the fixed point; the interior supplies the physical realization of the code; the horizon enforces the domain restriction D_O needed for the nontriviality of Prov_{D_O}. □

4.5 Many Underdetermined States versus Gödel-Fixed-Point States

It is important to separate two facts:

(A) Geometric underdetermination: there are continuum-many admissible global states sharing the same exterior restriction ω_{ext}.

(B) Gödelian fixed points: there are countably many diagonal fixed-point sentences (hence countably many "Gödel-type" codes n^*).

Conclusion (a) of the Master Theorem (geometric underdetermination) is operational, while conclusion (b) (Gödelian undecidability) is the specifically logical strengthening. The former motivates "incompleteness as underdetermination"; the latter justifies the "Gödelian structural parallel" claim.

4.6 Continuum of Exterior-Indistinguishable Admissible States

Theorem 4.10 (Exterior indistinguishability classes have continuum cardinality). *Fix an exterior restriction ω_{ext} on Obs_{D_O}. Suppose there exist at least two distinct admissible global extensions $\omega_1 \neq \omega_2$ with $\omega_i|_{\mathrm{Obs}_{D_O}} = \omega_{\mathrm{ext}}$. Then the set of admissible global extensions with the same exterior restriction has cardinality at least $2^{\aleph_0}$.*

Proof. The conclusion follows from Theorem 3.6 by either of the two methods established there: convex mixing within Class II (mixed admissible states), or the continuous family of pure Class I coherent excitations parameterized by $\alpha \in \mathbb{C}$ with $\mathrm{supp}(f) \subset R_{\mathrm{int}}$. In both cases, one obtains at least $2^{\aleph_0}$ admissible global extensions of ω_{ext}. □

Remark 4.11 (Admissibility of convex mixtures). The proof uses that convex mixtures of admissible states remain admissible. This holds for the following reasons: (i) The Hadamard condition is preserved under convex combinations, since the two-point function of a convex mixture $\omega_\lambda = \lambda\,\omega_1 + (1-\lambda)\,\omega_2$ is $\lambda\,W_{\omega_1} + (1-\lambda)\,W_{\omega_2}$, and the Hadamard singularity structure is stable under such combinations. (ii) Backreaction control is preserved because the renormalized stress-energy is linear in the state: $\langle T_{\mu\nu}\rangle^{\mathrm{ren}}_{\omega_\lambda} = \lambda\,\langle T_{\mu\nu}\rangle^{\mathrm{ren}}_{\omega_1} + (1-\lambda)\,\langle T_{\mu\nu}\rangle^{\mathrm{ren}}_{\omega_2}$, so $\delta g/g$ for the mixture is bounded by $\max(\delta g/g\,|_{\omega_1}, \delta g/g\,|_{\omega_2})$. (iii) Compatibility with exterior macroscopic data (condition (iv) of Definition 3.5) is preserved because exterior expectation values are linear and the two states agree on Obs_{D_O} by hypothesis.

4.7 Explicit Example: Solar-Mass Schwarzschild Black Hole

Parameters. For $M = M_\odot$: $r_{\mathrm{s}} = 2GM/c^2 \approx 2.95 \times 10^3\,\mathrm{m}$.

Entropy and capacity. $S_{\mathrm{BH}}/k_{\mathrm{B}} = 4\pi G M^2/(\hbar c) \approx 1.05 \times 10^{77}$, so $N_{\max} \sim \exp(1.05 \times 10^{77})$ and $\log_{10}(N_{\max}) \approx 4.56 \times 10^{76}$.

Backreaction scale for an interior encoding. Assume an interior encoding using n_q quanta of characteristic frequency $\omega \sim c/r_{\mathrm{s}}$:

$$\omega \approx 1.02 \times 10^5\,\mathrm{s}^{-1}\,, \qquad \varepsilon = \hbar\omega \approx 1.07 \times 10^{-29}\,\mathrm{J}\,, \qquad E_{\mathrm{enc}} \sim n_q\,\varepsilon\,.$$

The fractional ADM-energy backreaction is

$$\frac{E_{\text{enc}}}{Mc^2} \approx n_q \times 6.0 \times 10^{-77}.$$

For $n_q = 10^6$: $E_{\text{enc}}/(Mc^2) \approx 6.0 \times 10^{-71}$ (negligible).

Hawking temperature and evaporation time. $T_H = \hbar c^3/(8\pi G M k_B) \approx 6.17 \times 10^{-8}\,\text{K}$. The standard evaporation time (including the numerical prefactor) is $\tau_{\text{evap}} \approx 5120\pi\, G^2 M^3/(\hbar c^4) \approx 2.1 \times 10^{67}$ years.

Conclusion. All relevant scales confirm that for macroscopic black holes and any reasonable n_q well below $\exp(S_{\text{BH}}/k_B)$, the encoding backreaction is parametrically tiny and the semiclassical approximation is self-consistent on the exterior domain.

4.8 Comparison with Gödel's Original Construction

Gödel: $G \leftrightarrow \neg\text{Prov}(\ulcorner G \urcorner)$, hence (under standard hypotheses) $\text{PA} \nvdash G$ and $\text{PA} \nvdash \neg G$.

Here: $G^* \leftrightarrow \neg\text{Prov}_{D_O}(\ulcorner G^* \urcorner)$, hence (under Assumptions A–B) $\Phi_{D_O} \nvdash G^*$ and $\Phi_{D_O} \nvdash \neg G^*$.

The specifically physical ingredient is that Prov_{D_O} is domain-restricted by causal structure (the horizon), and the code $\ulcorner G^* \urcorner$ is physically realized by admissible interior degrees of freedom.

4.9 Equivalence: Incompleteness as Geometric Indistinguishability

Define exterior indistinguishability at level N by: $n \sim_{D_O,N} m :\equiv \text{SameExt}_N(n,m)$.

Theorem 4.12 (Geometric indistinguishability implies failure of exterior uniqueness)**.** *If there exist admissible codes $n \neq m$ with* $\text{SameExt}_N(n,m)$*, then neither n nor m satisfies* $\text{Desc}_{D_O,N}$.

Proof. If $\text{Desc}_{D_O,N}(n)$ held, then $\text{SameExt}_N(n,m)$ would imply $m = n$, contradiction. Similarly for m. □

Theorem 4.13 (Exterior underdetermination implies descriptive incompleteness)**.** *If the exterior restriction map $\omega \mapsto \omega|_{\text{Obs}_{D_O}}$ admits at least two distinct admissible extensions for some exterior restriction ω_{ext}, then Φ_{D_O} cannot provide a unique global-state description from exterior data alone, at the corresponding coding level N.*

This is the precise formal content of This is the precise formal content of the equivalence

$$\text{Geometric Indistinguishability} \iff \text{Domain-Relative Descriptive Incompleteness}$$

at the level of the coding predicates introduced in §4.1.

4.10 Observer-Relativity (Domain Dependence)

All results above are explicitly relative to the chosen causal domain D_O. Changing the domain (e.g. considering an infalling observer's causal past before the singularity) changes which observables belong to Obs_D and hence changes SameExt_N and $\text{Desc}_{D,N}$. This is the correct sense in which "completeness" is domain-relative.

4.11 Persistence Under Extensions That Do Not Change Causal Access

Any consistent extension of Φ_{D_O} that leaves the observational domain Obs_{D_O} unchanged (i.e., does not add new causal access to interior-supported observables) cannot restore exterior uniqueness. The underdetermination is enforced by the restriction map $\omega \mapsto \omega|_{\text{Obs}_{D_O}}$, not by the specific choice of axioms, provided those axioms remain confined to Obs_{D_O}. This is the structural counterpart, at the level of Φ_{D_O}, of the constraint theorem in §9.4: any extension that does not modify Obs_{D_O} falls within the framework's antecedent and remains incomplete.

4.12 Master Incompleteness Theorem

The following theorem consolidates the three principal results of this section into a single unified statement, collects all hypotheses explicitly, and clarifies the logical dependencies between the geometric, Gödelian, and categorical components of the argument.

Theorem 4.14 (Master Incompleteness Theorem for Semiclassical Gravity)**.** *Let (M,g) be a Schwarzschild spacetime with $M \gg m_P$, and let D_O denote the exterior domain of outer communication* (DOC)*. Let Φ_{D_O} be the restriction of semiclassical gravity to observables supported in Obs_{D_O}. Assume:*

- (H1) Recursive axiomatizability with arithmetic interpretation. *Φ_{D_O} is consistently and recursively axiomatizable and supports an interpretation of Robinson Arithmetic Q via multi-mode Fock space (Appendix E).*
- (H2) Split property. *The split property holds for the buffered exterior/interior algebra pair $(\mathfrak{A}_{\text{ext}}, \mathfrak{A}_{\text{int}})$ with buffer $\delta \gg \ell_P$ (Lemma B.1).*
- (H3) Consistency. *Φ_{D_O} is consistent.*
- (H4) Σ_1-soundness. *Φ_{D_O} does not prove false Σ_1-statements about coded exterior predictions at the fixed resolution level N.*

Then the following three conclusions hold jointly.

(a) Geometric underdetermination (C1). *The restriction map $\omega \mapsto \omega|_{\text{Obs}_{D_O}}$ is non-injective on the admissible state class: there exist distinct admissible states $\omega_1 \neq \omega_2$ with $\omega_1|_{\text{Obs}_{D_O}} = \omega_2|_{\text{Obs}_{D_O}}$. Moreover, the indistinguishability class has continuum cardinality,*

$$\left|\{\, \omega : \omega|_{\text{Obs}_{D_O}} = \omega_1|_{\text{Obs}_{D_O}} \,\}\right| \geq 2^{\aleph_0}.$$

This conclusion requires only (H2).

(b) Gödelian undecidability (C2). *There exists an admissible interior state s^* with gauge-invariant code $\gamma_N(s^*) = n^*$, where $n^* = \ulcorner G^* \urcorner$ is the diagonal fixed point of §4.1, such that*

$$\Phi_{D_O} \nvdash \text{Desc}_{D_O,N}(n^*) \qquad \text{and} \qquad \Phi_{D_O} \nvdash \neg\,\text{Desc}_{D_O,N}(n^*). \tag{21}$$

In particular, $\Phi_{D_O} \nvdash G^*$ and $\Phi_{D_O} \nvdash \neg G^*$. This conclusion requires all four hypotheses (H1)–(H4).

(c) Categorical equivalence. *Geometric indistinguishability (a) and domain-relative descriptive incompleteness are isomorphic instances of the Lawvere fixed-point obstruction (Proposition 8.3) under the encoding* γ_N*:*

$$\text{Geometric Indistinguishability} \cong \text{Domain-Relative Descriptive Incompleteness} \cong \text{Lawvere Fixed-Point Obstruction}.$$

Backreaction self-consistency. *For an interior encoding using* n_q *quanta of characteristic frequency* $\omega \sim c/r_s$*, the fractional geometric backreaction satisfies*

$$\frac{\delta g}{g} \lesssim n_q \times \left(\frac{m_P}{M}\right)^2 \ll 1 \qquad \text{for all } n_q \ll \exp(S_{BH}/k_B),$$

ensuring that the causal barrier and the domain restriction D_O *persist at semiclassical resolution throughout the construction.*

Proof (assembly of prior results). Part (a). By (H2), Lemma B.1 provides the split-property decomposition. The Bridge Theorem (Theorem 2.12) then yields distinct admissible global states with identical exterior restriction, and Theorem 4.10 promotes this pair to a continuum-sized equivalence class.

Part (b). Hypothesis (H1) supplies the arithmetic interpretation required for the diagonal lemma (§4.1). The existence of an admissible state realizing the fixed-point code n^* follows from (H2) (via the Bridge Theorem) together with the Bekenstein–Hawking capacity bound of §3.4 (Theorem 4.3). Undecidability of G^* in Φ_{D_O} then follows from (H3) and (H4) by the standard Gödel argument (Theorem 4.4), and non-derivability of $\mathrm{Desc}_{D_O,N}(n^*)$ is Corollary 4.5.

Part (c). We prove both directions of the equivalence

$$\text{Geometric Indistinguishability} \iff \text{Domain-Relative Descriptive Incompleteness}$$

Forward direction (indistinguishability ⇒ incompleteness). By Theorem 4.12: if there exist distinct admissible codes $n \neq m$ satisfying $\mathrm{SameExt}_N(n, m)$, then neither code satisfies $\mathrm{Desc}_{D_O,N}$, so Φ_{D_O} cannot derive a unique global-state description from exterior data alone.

Reverse direction (incompleteness ⇒ indistinguishability). Suppose Φ_{D_O} is descriptively incomplete: there exists an admissible global state s such that no exterior-derivable sentence uniquely singles out $[s]_{D_O}$ within $\mathcal{S}_{\mathrm{adm}}$. By Definition 2.20(iv)–(v), this failure is of *causal* origin if and only if it is lifted when the observational domain is enlarged to $D' \supset D_O$ with $R \subset J^-(D')$: an observer with causal access to the interior can distinguish states that are exterior-equivalent. Under the locality axiom of Φ_{D_O} (Definition 2.22), any derivation grounded solely in Obs_{D_O} is blind to interior-supported observables; hence if the incompleteness is causal (Definition 2.20(v)), there must exist two distinct admissible states with identical exterior restriction, i.e., geometric indistinguishability holds (Theorem 4.13).

Lawvere identification. Both the forward and reverse directions are instances of the categorical fixed-point obstruction: the non-surjectivity of the evaluation map $\hat{\phi}$ is equivalent (via the contrapositive of Lawvere's theorem) to the existence of a fixed-point-free endomorphism on the codomain, which is precisely the negation map on $\{0, 1\}$ identifying derivable/non-derivable sentences. The identification is made precise in Proposition 8.4.

Backreaction. Immediate from Theorem 3.9 with $\omega \sim c/r_s$ and Corollary 3.11. □

Remark 4.15 (Logical role of each hypothesis). The four hypotheses (H1)–(H4) play distinct roles in the proof of conclusions (a) and (b), as follows. They are not strictly independent in the sense of being separately variable while leaving the others intact, since (H1) provides the formal language in which (H3) and (H4) are even expressible. However, the following implications clarify their respective contributions.

Role of (H2) for conclusion (a). The split property is sufficient to establish geometric underdetermination: by Theorem 2.12, it guarantees the existence of distinct admissible global extensions of any exterior restriction, and Theorem 3.6 (via either Method 1 or Method 2 of its proof) promotes this to a continuum-sized indistinguishability class. Conclusion (a) thus requires (H2) and the underlying admissibility framework of Definition 3.5, but does not invoke arithmetic interpretation, consistency, or soundness in any essential way.

Role of (H1)–(H4) for conclusion (b). The Gödelian undecidability claim requires all four hypotheses, in distinct capacities:

(H1) supplies the arithmetic substrate (Robinson Arithmetic Q via Fock space) needed to formulate the diagonal lemma and the provability predicate Prov_{D_O}. Without (H1), the fixed-point sentence G^* cannot be constructed.

(H2) ensures that the domain restriction is nontrivial, i.e., that Prov_{D_O} genuinely differs from $\mathrm{Prov}_{\Phi_{\mathrm{semi}}}$. Without (H2), the exterior theory might already determine global states, rendering the diagonal construction vacuous.

(H3) (consistency) is required for the first half of Theorem 4.4: $\Phi_{D_O} \nvdash G^*$.

(H4) (Σ_1-soundness) is required for the second half: $\Phi_{D_O} \nvdash \neg G^*$. Together with the HBL conditions (Assumption C), it ensures that derivability of $\mathrm{Prov}_{D_O}(\ulcorner G^* \urcorner)$ entails the existence of an actual proof.

Modifications that evade the conclusions. A theory that violates (H2)—for instance, by replacing the semiclassical interior sector with a fundamentally discrete Hilbert space, or by enlarging the accessible algebra beyond Obs_{D_O} via holographic reconstruction—may evade conclusion (a) and consequently (b). This is the precise sense in which the result functions as a structural constraint on completion strategies: any theory restoring global describability must modify at least one of (H1)–(H4), as elaborated in Section 5.

4.13 Summary of Main Results

The principal results of this section may be summarized as follows:

(i) We defined a domain-relative uniqueness predicate $\mathrm{Desc}_{D_O,N}(n)$ grounded in exterior observables (§4.1).

(ii) Using arithmetic interpretation and the diagonal lemma, we constructed a fixed-point sentence G^* and a physically admissible coded state s^* with $\gamma_N(s^*) = \ulcorner G^* \urcorner$ (§4.2, Theorem 4.3).

(iii) Under hypotheses (H3)–(H4) of the Master Theorem, G^* is undecidable in Φ_{D_O} (Theorem 4.4).

(iv) Independently, causal barriers imply exterior underdetermination: there exist continuum-many admissible global states with identical exterior restriction (Theorem 3.6).

(v) All three facts are unified in the Master Incompleteness Theorem (Theorem 4.14), which establishes that semiclassical gravity, when restricted to an exterior causal domain, exhibits a structural limitation on descriptive completeness enforced by causal geometry.

5 Objections and Responses

This section addresses standard objections to our domain-relative incompleteness result. Throughout, we keep two claims distinct:

(C1) Geometric underdetermination (operational): for an exterior causal domain D_O, the restriction map $\omega \mapsto \omega|_{\mathrm{Obs}_{D_O}}$ is not injective on the admissible state class; hence there exist distinct admissible global states with identical exterior predictions.

(C2) Gödel-type undecidability (logical): once Φ_{D_O} is formalized so that the coding-interface predicates (admissibility, exterior agreement, and domain-relative uniqueness $\mathrm{Desc}_{D_O,N}$) are representable, diagonalization yields a fixed-point sentence G^* that is undecidable in Φ_{D_O} under standard hypotheses (consistency plus a mild soundness/1-consistency condition).

Objections typically conflate (C1) and (C2), or interpret them as claims about "information loss" or "measurement limits." We respond accordingly.

5.1 Objection 1: Holographic Quantum Gravity

Objection. "Holography encodes all bulk information on the boundary. Therefore interior states are completely describable from the exterior, contradicting your incompleteness claim."

Response (scope and observables). Our theorem is explicitly about semiclassical gravity restricted to an exterior causal domain D_O and to the corresponding semiclassical observable content Obs_{D_O}. Holographic duality—when available—is a statement about a different theory with a different notion of fundamental degrees of freedom and, crucially, a different notion of what counts as an admissible observable.

(1) Domain-of-applicability separation. In AdS/CFT, the "complete" description (if one accepts exact duality) is furnished by the boundary CFT, not by the semiclassical bulk effective field theory restricted to exterior local observables. The objection targets completeness of the underlying quantum-gravitational description; our claim targets incompleteness of the exterior-restricted semiclassical description.

(2) Nonlocal decoding versus semiclassical restriction. Even in AdS/CFT, reconstructing an interior bulk operator from boundary data typically involves highly nonlocal boundary operators (or state-dependent reconstruction procedures). Such decoding operations are not elements of the semiclassical exterior algebra Obs_{D_O} (nor are they generally definable as local bulk observables supported in D_O). Hence they lie outside the formal and operational scope of Φ_{D_O}.

(3) Asymptotically flat case. Our core construction is formulated for asymptotically flat black holes with exterior domain given by the DOC relative to $\mathscr{I}^+$. A fully established, exact holographic dual for generic asymptotically flat spacetimes is not currently on the same footing as AdS/CFT. Even granting a future flat-space holography, the same scope distinction applies.

Conclusion. Holography (where exact) is best interpreted as a candidate mechanism by which a more fundamental theory can evade or resolve the semiclassical domain-relative incompleteness. This is consistent with, not contradictory to, our result. The recent replica-wormhole and quantum-extremal-surface constructions (Penington [43], Engelhardt–Wall [44], Almheiri *et al.* [45]) provide a concrete example of how the strict locality structure of Φ_{D_O} is modified to restore exterior injectivity at the level of the Page curve; we discuss this further in §9.2.

5.2 Objection 2: Observer-Dependence Implies Subjectivity

Objection. "Incompleteness is observer-relative. Does that reduce the claim to an epistemic or subjective limitation?"

Response (geometric objectivity). Observer-relativity here is not arbitrariness; it is dependence on a geometrically well-defined causal domain.

(1) The relevant invariant. For each operational domain D_O, the algebra Obs_{D_O} is determined by causal structure. The statements "$R \cap J^-(D_O) = \varnothing$" and "$\omega \mapsto \omega|_{\mathrm{Obs}_{D_O}}$ is non-injective" are objective facts about (M, g) and the chosen domain.

(2) Analogy with special relativity. Frame-dependence in SR does not imply subjectivity; it reflects an invariant geometric structure (the Lorentzian metric). Likewise, "domain-relative describability" depends on the invariant causal structure (the conformal class $[g_{\mu\nu}]$ and the

induced causal relations), not on the observer's beliefs, technology, or computational resources.

Conclusion. The incompleteness is ontologically grounded in causal geometry. "Relative" means "domain-indexed by causal structure," not "subjective."

5.3 Objection 3: Hawking Radiation and Information Recovery

Objection. "Hawking evaporation releases information. Therefore interior states become observable, and incompleteness is only temporary."

Response (what the theorem actually quantifies). *(1) Regime statement.* Our theorem is formulated within the semiclassical regime ($M \gg m_{\mathrm{P}}$) and with a fixed exterior causal domain D_O in which an event horizon functions as an absolute barrier for interior-supported observables. In that regime, the restriction map $\omega \mapsto \omega|_{\mathrm{Obs}_{D_O}}$ is non-injective; this is not a timescale claim but a structural claim about causal access and observable algebras.

(2) Timescale realism. For $M \sim M_\odot$, the standard evaporation time is $\tau_{\mathrm{evap}} \sim 10^{67}$ years. Even if one believes full unitarity and eventual purification of the radiation in a complete theory, the semiclassical black hole persists for essentially the entire cosmological timescale. Thus the horizon-induced domain restriction remains physically relevant for any conceivable observational program.

(3) Semiclassical versus quantum-gravitational mechanisms. Any genuine recovery of interior information in the Hawking radiation requires physics beyond the naive semiclassical treatment. Modern proposals (including nonperturbative contributions, "islands," or other quantum-gravity ingredients) precisely modify the effective semiclassical description in ways that do not remain confined to the original semiclassical exterior algebra Obs_{D_O}. This aligns with our scope claim: completeness, if restored, is restored by transcending the exterior-restricted semiclassical framework.

Conclusion. Evaporation does not refute the claim "Φ_{D_O} is incomplete within its semiclassical domain"; it motivates the expectation that a more fundamental theory changes the relevant notion of observables and/or causal structure.

5.4 Objection 4: Quantum Gravity Resolves Incompleteness

Objection. "Quantum gravity proposals (string theory, LQG, etc.) modify horizons or interior structure and thereby resolve incompleteness."

Response (compatibility and a necessary-condition reading). We agree with the conditional form: if a theory achieves global descriptive completeness, it must evade at least one of the four hypotheses (H1)–(H4) of the Master Theorem (§4.12). The corresponding modifications are:

- *(i)* modification of the strict causal barrier relevant to D_O or of the admissible observable content (effectively a violation of (H2): the split property no longer enforces independence between Obs_{D_O} and interior degrees of freedom);
- *(ii)* modification of the admissible state space, e.g., replacing an effectively infinite-dimensional interior sector by a finite or fundamentally discrete structure at the relevant resolution (weakening the cardinality assumption underlying conclusion (a) of the Master Theorem);
- *(iii)* modification of the arithmetic representability hypothesis (H1): if the interior sector cannot, even in principle, support an interpretation of Robinson Arithmetic, then the diagonal construction does not apply.

A detailed examination of these escape routes is given in §9.4. Many quantum-gravity proposals implement one or more of (i)–(iii). Our contribution is not to adjudicate among these proposals, but to formalize why some such modification is required if one demands global completeness.

5.5 Objection 5: Black Hole Complementarity

Objection. "Complementarity says interior and exterior descriptions are complementary and each is complete in its domain. Hence your incompleteness is irrelevant."

Response (compatibility once 'completeness' is domain-indexed). Complementarity is broadly compatible with our framework when formulated in algebraic/domain terms.

(1) What complementarity addresses. Complementarity is primarily about consistency of descriptions across observers with different operational algebras (exterior vs infalling). It does not, by itself, guarantee that the exterior algebra Obs_{D_O} determines a unique global state, i.e., it does not assert injectivity of $\omega \mapsto \omega|_{\mathrm{Obs}_{D_O}}$.

(2) What our incompleteness addresses. Our claim is that, within semiclassical gravity and for an exterior domain D_O, the observable content accessible in D_O underdetermines the global state. This is a statement about non-injectivity of restriction and the failure of domain-relative uniqueness predicates—not a claim that exterior predictions for exterior observables are internally inconsistent.

(3) When tension could arise. If one strengthens complementarity into a claim of a single globally complete description derivable entirely from exterior semiclassical observables, then it conflicts with (C1). But standard complementarity need not make that strengthened

claim; rather, it typically posits that no single observer has simultaneous access to both complementary algebras.

Conclusion. Complementarity, properly stated as domain-indexed descriptive adequacy, is consistent with our result. If complementarity is upgraded to a global reconstruction principle, it requires additional non-semiclassical structure (consistent with §5.1–5.4).

5.6 Objection 6: Computational Complexity versus Logical Incompleteness

Objection. "You are describing computational intractability, not Gödelian underivability."

Response (insufficiency of data versus difficulty of computation). *(1) Complexity claims:* a statement is decidable in principle but computationally infeasible.

(2) Our geometric claim (C1): even with unlimited computation, if the accessible data (the full set of exterior expectation values in Obs_{D_O}, at the chosen resolution) is compatible with multiple admissible global states, then no algorithm can output a unique global state without additional input. This is not "hard"; it is underdetermined.

(3) Our logical claim (C2): given representability of the predicates Ad_N, $\mathrm{SameExt}_N$, and $\mathrm{Desc}_{D_O,N}$ (§4.1), diagonalization yields an undecidable sentence G^* internal to Φ_{D_O}. This is a statement about the absence of derivations in Φ_{D_O}, not about the length or computational cost of derivations.

Conclusion. The obstruction is structural (underdetermination enforced by causal restriction) and, when formalized, yields genuine Gödel-type undecidability rather than a complexity barrier.

5.7 Objection 7: Conflation with the Quantum Measurement Problem

Objection. "This is just the measurement problem: the interior is in superposition until measured."

Response (distinct logical structure). The quantum measurement problem concerns state update/interpretation under measurement and the status of superposition relative to outcomes. Our claim concerns non-injectivity of the restriction map to a causally limited observable algebra.

Even if one assumes idealized complete measurement of all observables in Obs_{D_O} (or of a separating family dense in Obs_{D_O}), the global state remains underdetermined because interior-supported observables are excluded by causal structure. By Theorem 3.6, the multiplicity of compatible global states has cardinality at least $2^{\aleph_0}$; this multiplicity is not a "superposition waiting to collapse" but a structural equivalence class induced by causal restriction, which persists regardless of the interpretation of measurement chosen.

5.8 What Our Theorem Does NOT Claim

1. We do not claim that Gödel's first incompleteness theorem "applies directly" to physics without additional representability assumptions. We claim that semiclassical gravity, when formalized with a domain-restricted provability notion and a coding interface, can instantiate Gödel-type diagonal fixed points and undecidability.

2. We do not claim that all physical theories are incomplete. We claim a precise incompleteness for the exterior-restricted semiclassical framework under the stated conditions; more fundamental theories may evade it by changing observables, state space, or causal structure.

3. We do not claim "information destruction" as a theorem. We claim that semiclassical exterior observables underdetermine interior microstructure; whether information is globally preserved is a question for the underlying completion.

4. We do not claim observer-independence of completeness. We claim domain-indexed completeness/incompleteness determined by the objective causal structure and the corresponding observable algebra.

5. We do not claim that the interior is "unknowable" in an absolute sense. We claim that it is not uniquely describable from the exterior causal domain within the semiclassical observable content.

6. We do not claim to prove quantum gravity must exist. We claim that if one demands global descriptive completeness for black holes, then one must transcend at least one semiclassical premise (causal barrier, observable content, admissible state space, or coding-interface representability).

6 Generalizations

This section shows that the incompleteness mechanism is not specific to Schwarzschild geometry. The core input is domain restriction by a causal barrier together with a nontrivial algebra of degrees of freedom outside the observer's causal past. Accordingly, we formulate the results uniformly in terms of an exterior causal domain D_O and the associated algebra of accessible observables $\mathfrak{A}(D_O)$. (Notation: Φ_{D_O} denotes the semiclassical framework restricted to $\mathfrak{A}(D_O)$; "global descriptive completeness relative to D_O" means uniqueness of the underlying physical state given all predictions on $\mathfrak{A}(D_O)$, at the stipulated physical resolution.)

Conventions. In §6.1–6.5 we use geometrized units $G = c = 1$ for compactness. Standard factors of G and c can be reinstated by dimensional analysis. Entropy formulas are written with k_{B} and $\hbar$ explicit when needed.

6.1 Rotating Black Holes (Kerr)

The Kerr family (Boyer–Lindquist coordinates) is

$$\begin{aligned} ds^2 = &-\Big(1 - \frac{2Mr}{\Sigma}\Big)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,dt\,d\varphi \\ &+ \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 \\ &+ \sin^2\theta\,\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\,d\varphi^2\,, \end{aligned} \tag{22}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, and $a := J/M$.

Event horizon. For subextremal Kerr ($|a| < M$) the outer event horizon is at $r_+ = M + \sqrt{M^2 - a^2}$. The exterior domain for observers at $\mathscr{I}^+$ is the DOC, which lies in $r > r_+$.

Theorem 6.1 (Kerr domain-relative incompleteness)**.** *For any subextremal Kerr black hole ($|a| < M$), the exterior-restricted semiclassical theory Φ_{D_O} with $D_O =$* DOC *is descriptively incomplete relative to the full global state space: there exist distinct admissible global states whose restrictions to $\mathfrak{A}(D_O)$ coincide.*

Proof sketch. The event horizon $\mathcal{H} = \partial J^-(\mathscr{I}^+)$ is a null causal barrier. Degrees of freedom supported in the interior region cannot affect expectation values of observables in $\mathfrak{A}(D_O)$. Under standard locality/microcausality assumptions for the net of local algebras, distinct global states may agree on $\mathfrak{A}(D_O)$, hence Φ_{D_O} cannot uniquely fix the global state. The argument is the same as in the Schwarzschild case, replacing r_{s} by r_+ and D_O by the Kerr DOC. □

Entropy and capacity. The Bekenstein–Hawking entropy is $S_{\mathrm{BH}} = k_{\mathrm{B}}A_+/(4\,\ell_{\mathrm{P}}^2)$, with $A_+ = 4\pi(r_+^2 + a^2)$. For fixed M, S_{BH} decreases with spin but remains enormous for astrophysical masses, so the encoding-capacity estimates used in Sections 3–4 remain parametrically unchanged.

Remark 6.2 (Ergosphere)*.* The ergoregion (between the stationary-limit surface and the horizon) affects the kinematics of stationary observers but does not alter the existence of the causal barrier at $r = r_+$, which is the only geometric input needed for domain-relative underdetermination.

6.2 Charged Black Holes (Reissner–Nordström)

The Reissner–Nordström (RN) metric is

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2\,d\Omega^2\,, \qquad f(r) = 1 - \tfrac{2M}{r} + \tfrac{\tilde{Q}^2}{r^2}\,, \tag{23}$$

where $\tilde{Q}^2$ is the geometrized charge parameter.

Horizons. For subextremal RN ($\tilde{Q}^2 < M^2$), the outer and inner horizons are $r_\pm = M \pm \sqrt{M^2 - \tilde{Q}^2}$, with r_+ serving as the event horizon.

Theorem 6.3 (RN domain-relative incompleteness)**.** *For any subextremal RN black hole ($\tilde{Q}^2 < M^2$), the exterior-restricted semiclassical theory Φ_{D_O} on the RN DOC is descriptively incomplete relative to global states.*

Proof sketch. The outer event horizon is a causal barrier. As in §6.1, locality implies that distinct admissible global states can agree on all observables in $\mathfrak{A}(D_O)$. The presence of an inner horizon (r_-) is not required for the argument and plays no essential role in the exterior incompleteness claim. □

Caveat (inner-horizon instability). Classical and semiclassical analyses indicate potential instabilities near the Cauchy horizon (mass inflation). This does not affect the exterior-restricted conclusion, which depends only on the existence of the outer event horizon and the restriction to $\mathfrak{A}(D_O)$.

6.2.1 The Quantum-Gravity Objection (Contextualized)

Objection. "These results are semiclassical; a full quantum-gravity completion might restore completeness."

Response (scope). This is precisely the scope distinction emphasized in §5.4 (and unpacked structurally in §9.4). The claims of the present section concern Φ_{D_O} as an exterior-restricted semiclassical framework. A quantum-gravity completion may evade incompleteness only by violating one of the structural ingredients of the Master Theorem (Theorem 4.14): the causal barrier (H2), the cardinality assumption underlying conclusion (a), or the arithmetic representability (H1). The semiclassical result is nontrivial because it identifies which structural inputs force underdetermination, thereby delimiting what any completion must modify to achieve global descriptive completeness.

6.3 Cosmological Horizons (de Sitter)

In the static patch of de Sitter space ($\Lambda > 0$),

$$ds^2 = -\Big(1 - \frac{\Lambda r^2}{3}\Big)dt^2 + \Big(1 - \frac{\Lambda r^2}{3}\Big)^{-1}dr^2 + r^2\,d\Omega^2\,. \tag{24}$$

The cosmological horizon for the static observer is at $r_c = \sqrt{3/\Lambda}$.

Theorem 6.4 (Cosmological domain-relative incompleteness)**.** *For a timelike geodesic observer O in de Sitter space, let D_O be the causal diamond associated with O (equivalently, the static patch). Then the semiclassical theory restricted to $\mathfrak{A}(D_O)$ is descriptively incomplete relative to global states: there exist distinct admissible global states that agree on all observables accessible within D_O.*

Proof sketch. The cosmological horizon is the boundary of the causal past of O. Regions outside D_O are causally inaccessible, and local observables supported there cannot influence expectation values in $\mathfrak{A}(D_O)$. Under standard locality assumptions, restriction to $\mathfrak{A}(D_O)$ is not injective on admissible global states. □

Entropy. The Gibbons–Hawking entropy of the de Sitter horizon, $S_{\rm GH} = k_{\rm B} A_c/(4\,\ell_{\rm P}^2)$ with $A_c = 4\pi r_c^2$, provides the same "finite-area, vast-capacity" structure as in the black-hole case.

6.4 Rindler Horizons (Accelerated Observers)

An observer with constant proper acceleration a in Minkowski spacetime is naturally associated with a Rindler wedge. In Rindler coordinates (η, ξ, y, z) on the right wedge $(\xi > 0)$,

$$ds^2 = -(a\xi)^2\, d\eta^2 + d\xi^2 + dy^2 + dz^2\,, \tag{25}$$

and $\xi = 0$ is a (coordinate) horizon corresponding to a null boundary of $J^-(O_R)$.

Theorem 6.5 (Rindler domain-relative incompleteness)**.** *For a uniformly accelerated observer O_R, the restriction of the (otherwise complete) Minkowski QFT to the Rindler wedge algebra $\mathfrak{A}(D_{O_R})$ is descriptively incomplete relative to global states.*

Proof sketch. The Rindler horizon is a causal barrier relative to O_R. Distinct global states can agree on the wedge algebra (and hence on all observables accessible to O_R), so $\Phi_{D_{O_R}}$ does not uniquely determine the global state. □

Remark 6.6 (Observer dependence)*.* Here the causal barrier is observer-relative: inertial observers have no such horizon. This does not make the claim subjective; it means incompleteness is indexed by the operational domain D_{O_R} determined by the observer's worldline and the ambient causal structure.

Unruh effect. The Unruh temperature $T_U = \hbar a/(2\pi k_{\rm B} c)$ is a thermal manifestation of the wedge restriction, conceptually analogous to Hawking thermality in the black-hole case.

The Unruh temperature can also be derived from the Bisognano–Wichmann theorem [50] via modular theory, providing an independent algebraic foundation for wedge-restricted thermality. Our incompleteness argument, however, requires only locality and the split property, not the full modular apparatus.

6.5 Higher-Dimensional Black Holes

In d spacetime dimensions $(d \geq 4)$, the Schwarzschild–Tangherlini metric reads

$$ds^2 = -\Big(1 - \frac{\mu}{r^{d-3}}\Big)dt^2 + \Big(1 - \frac{\mu}{r^{d-3}}\Big)^{-1} dr^2 + r^2\, d\Omega_{d-2}^2\,, \tag{26}$$

where μ is proportional to the mass and $d\Omega_{d-2}^2$ is the round metric on S^{d-2}. The event horizon is at $r = r_s$ with $r_s^{d-3} = \mu$.

Theorem 6.7 (Higher-dimensional domain-relative incompleteness)**.** *For $d \geq 4$, Schwarzschild–Tangherlini black holes exhibit the same exterior domain-relative descriptive incompleteness: restricting a semiclassical theory to the DOC algebra $\mathfrak{A}(D_O)$ does not uniquely fix the global state.*

Proof sketch. The argument depends only on the existence of an event horizon as a causal barrier and on locality; it is insensitive to dimension. □

Entropy scaling. $S_{\rm BH} \propto A_{d-2}/(4\,\ell_{\rm P}^{d-2})$, and A_{d-2} grows as r_s^{d-2}, again yielding vast encoding capacity for macroscopic horizons.

6.6 Analog Horizon Systems

Analog gravity systems (e.g., acoustic horizons in fluids or Bose–Einstein condensates) can exhibit an effective causal barrier for perturbations.

Example (1+1 dimensional acoustic metric). For a barotropic, irrotational flow with velocity $v(x)$ and sound speed c_s, linearized phonons propagate in an effective metric (up to a conformal factor)

$$ds^2_{\rm eff} \;\propto\; -c_s^2\, dt^2 + (dx - v(x)\, dt)^2\,.$$

An acoustic horizon forms where $|v(x)| = c_s$, separating subsonic and supersonic regions for phonon propagation.

Theorem 6.8 (Analog domain-relative incompleteness; structural form)**.** *If an analog system admits (i) an effective horizon for the relevant excitations and (ii) a local quantum field description of those excitations on the effective background, then restricting to the exterior-accessible phonon algebra yields descriptive underdetermination of "interior" phonon configurations, in the same structural sense as Sections 4 and 6.1–6.4.*

Proof sketch. Replace the gravitational causal barrier by the effective horizon barrier for the relevant signal cones (phonon cones). Locality of the effective field theory then implies that observables supported beyond the analog horizon cannot influence exterior observables, yielding non-injectivity of restriction to the exterior algebra. □

Remark 6.9 (Structural realization)*.* Analog systems realize the minimal structural inputs of Theorem 6.10 (effective horizon together with local field degrees of freedom on the effective background) in a substrate distinct from gravitational black holes. Their relevance for the present framework is therefore structural rather than empirical: they instantiate the same algebraic mechanism on a different effective background, confirming that the incompleteness pattern depends on the abstract structure (causal barrier + locality + nontrivial inaccessible degrees) and not on specifically gravitational features.

6.7 The Minimal Incompleteness Criterion

The earlier sections can be summarized as a general algebraic criterion.

Theorem 6.10 (Minimal structural criterion)**.** *Let $\mathfrak{A}$ be the global (quasi-local) observable algebra of a relativistic QFT on a spacetime (or effective spacetime) background, and let $\mathfrak{A}(D_O) \subset \mathfrak{A}$ be the subalgebra generated by observables localized in an observer's operational domain D_O. Suppose:*

(1) Causal barrier / proper inclusion: *$\mathfrak{A}(D_O)$ is a proper subalgebra of $\mathfrak{A}$ because there exists a nonempty region R with $R \cap J^-(D_O) = \varnothing$ and with a nontrivial local algebra $\mathfrak{A}(R)$ that is causally disjoint from $\mathfrak{A}(D_O)$ (in the sense of microcausality).*

(2) Nontrivial inaccessible degrees: *the state space on $\mathfrak{A}(R)$ is non-singleton (equivalently, $\mathfrak{A}(R)$ admits at least two distinct normal states compatible with the chosen admissibility class).*

(3) Split/independence: *the inclusion $\mathfrak{A}(D_O) \subset \mathfrak{A}$ admits sufficiently many product-type extensions so that exterior and inaccessible sectors can be independently varied at fixed exterior restriction.*

Then the restriction map

$$\mathrm{Res}\colon \mathrm{States}(\mathfrak{A}) \to \mathrm{States}(\mathfrak{A}(D_O))\,, \qquad \omega \mapsto \omega\big|_{\mathfrak{A}(D_O)}$$

is not injective on the admissible state class. Consequently, Φ_{D_O} is descriptively incomplete relative to global states.

The three structural ingredients identified by this theorem—causal barrier, nontrivial inaccessible degrees, and split/independence—are precisely the structural ingredients whose modification is required to restore operational completeness, as analyzed in §9.4.

Proof sketch. By (1)–(3), pick two distinct admissible states σ_1, σ_2 on $\mathfrak{A}(R)$ and fix an exterior state ω_{ext} on $\mathfrak{A}(D_O)$. Use the split/independence property to form two admissible global states ω_1, ω_2 extending ω_{ext} while differing on $\mathfrak{A}(R)$. By construction, ω_1 and ω_2 coincide on $\mathfrak{A}(D_O)$ yet are globally distinct. □

Remark 6.11.

(i) The criterion avoids questionable "countability of observables" assumptions: incompleteness is driven by proper-subalgebra restriction and nontrivial commutant/inaccessible sector, not by cardinality bookkeeping.

(ii) When combined with the representability/coding interface developed in Section 2, this operational underdetermination can be upgraded to Gödel-type undecidability statements (as in Section 4), provided the relevant predicates are representable in Φ_{D_O}.

6.8 Systems Where Incompleteness Fails

(1) No causal restriction (global domain). If $D_O = M$ (no causally inaccessible region for the operational domain under consideration), then $\mathfrak{A}(D_O) = \mathfrak{A}$ and the restriction map is trivially injective. This covers inertial observers in globally hyperbolic Minkowski space when one takes the operational domain to be all of M.

(2) Trivial inaccessible sector. If the inaccessible region R carries no independent degrees of freedom in the admissible class (e.g., $\mathfrak{A}(R)$ is effectively trivial under the imposed constraints), then the restriction may become injective.

(3) Completions that change the premises. A quantum-gravity completion may evade the criterion by modifying the causal barrier (so that $\mathfrak{A}(D_O)$ is no longer proper in the relevant sense), or by enlarging the admissible observable content beyond semiclassical locality (so that the "exterior" algebra is no longer limited to $\mathfrak{A}(D_O)$).

6.9 Unified Principle (Domain-Indexed Form)

Principle of causal-domain incompleteness. Predictive completeness on $\mathfrak{A}(D_O)$ (i.e., correct determination of all correlators of accessible observables) does not imply global descriptive completeness. Global descriptive completeness relative to D_O holds only when the operational domain is not causally restricted in a way that renders $\mathfrak{A}(D_O)$ a proper subalgebra with a nontrivial inaccessible complement.

Equivalently (under the structural criterion of §6.7):

$$\mathfrak{A}(D_O) = \mathfrak{A} \quad \Longleftrightarrow \quad \text{restriction is injective} \quad \Longleftrightarrow \quad \text{domain-relative descriptive completeness holds}\,.$$

6.10 Incompleteness as an Ontological Boundary

The preceding examples suggest that "incompleteness" is naturally associated with boundaries between operationally/causally separated domains.

Definition 6.12 (Ontological boundary; domain-relative form)**.** A hypersurface Σ is an *ontological boundary* relative to a domain D_1 if it induces a decomposition of operational domains D_1 and D_2 such that $\mathfrak{A}(D_1)$ is a proper subalgebra of the global algebra relevant to $D_1 \cup D_2$, and restriction to $\mathfrak{A}(D_1)$ is non-injective on the admissible state class due to degrees of freedom supported in D_2.

Conjecture 6.13 (Generalized domain-relative incompleteness)**.** *Whenever a physically admissible theory admits an ontological boundary Σ separating D_1 from D_2 in the above sense, the restriction Φ_{D_1} is descriptively incomplete with respect to states supported in (or differing on) D_2.*

Remark 6.14. This conjecture is not used in the proofs of the main results; it is a conceptual generalization aimed at isolating the common structure behind horizons (black-hole, cosmological, Rindler, and analog) and any other causal/structural separations that yield a proper-subalgebra restriction with independent degrees of freedom beyond the domain.

7 Implications for the Information Paradox and Quantum Gravity Programs

Throughout, we keep the scope fixed: the incompleteness theorems are statements about exterior-restricted semiclassical theories (QFT on a classical curved background with a causal barrier).

7.1 Connection to the Black Hole Information Problem

Hawking's semiclassical calculation implies that an exterior observer, who has access only to an exterior observable algebra $\mathfrak{A}_{\text{ext}}$, must describe the radiation state by a reduced density operator obtained by tracing over the causally inaccessible degrees of freedom. In the present framework, this is not an auxiliary interpretational move but a structural necessity: by Theorem 4.10 (geometric indistinguishability), there exist continuum-many interior completions compatible with any fixed exterior data set, hence no exterior-local derivation can select a unique global microstate.

Accordingly, the "information loss" diagnosed in semiclassical reasoning should be read as follows: the exterior description is not wrong in its own domain; rather, it is intrinsically non-injective as a map from global physical states to exterior data. The mixedness of the exterior description is the operational signature of this non-injectivity. In this sense, the information problem is recast from a purely dynamical puzzle ("how does unitary evolution manifest in the Hawking flux?") into a structural statement ("an exterior-local semiclassical theory cannot, even in principle, provide a unique global description when a causal barrier is present").

The connection to recent semiclassical extensions of this picture—in particular replica-wormhole and quantum-extremal-surface constructions—is taken up in §9.2, where it is shown that such extensions exemplify the kind of structural modification required to restore exterior injectivity, rather than contradicting the incompleteness theorems.

7.2 Implications for a "Theory of Everything"

The incompleteness results do not preclude a ToE understood as a globally defined dynamical law. They constrain, instead, what "completeness" can mean operationally for observers confined to a proper causal domain.

We distinguish three notions:

(1) Global completeness (law-level): a theory provides well-posed dynamics for global states on admissible spacetimes.

(2) Observer-relative completeness (operational): for a given observer O, the restriction Φ_O to Obs_O uniquely determines the physically realized state.

(3) Effective completeness (model-level): within a restricted regime of approximation and accessible observables, predictions are stable and exhaustive for those observables.

Our theorems rule out (2) whenever a causal barrier is present and the interior state space is nontrivial in the sense of Theorem 6.10. A ToE may still satisfy (1) while failing (2) for essentially geometric reasons. Thus, the correct lesson is not "no ToE is possible," but rather: any ToE that admits horizons will entail observer-relative incompleteness for observers whose causal past does not cover the entire spacetime.

7.3 Interfaces with Quantum-Gravity Programs

The structural content of the incompleteness theorems can be read in two complementary ways. From within Φ_{ext}, the theorems describe an irreducible obstruction. From the perspective of any candidate quantum-gravitational completion, however, the same theorems function as a *constraint*: any framework that aspires to operational completeness must alter the structural ingredients identified abstractly in §9.4. We here examine how three principal lines in the quantum-gravity literature relate to these structural ingredients, focusing on the substantive physics of each program rather than on its abstract categorization.

Holography and boundary dualities. In holographic frameworks (most concretely AdS/CFT), the bulk-local algebra is effectively replaced by boundary degrees of freedom that encode bulk data nonlocally. Bulk operators in the entanglement wedge of a boundary subregion are reconstructed via state-dependent or precursor-type constructions that are intrinsically nonlocal from the bulk perspective. From the standpoint of Φ_{ext}, holographic completion does not refute the incompleteness theorems—it bypasses them by working in a description for which the strict exterior/interior algebraic separation employed in our construction is not the fundamental locality structure. The applicability of these techniques to asymptotically flat black holes, where the present framework is most directly formulated, remains an open question; we discuss this in §7.5.

Fuzzball-type microstate geometries. Microstate-geometry programs propose that, at the fundamental level, the classical interior region and horizon of a Schwarzschild-like black hole are replaced by a family of horizonless geometries whose collective coarse-graining reproduces the macroscopic black-hole observables. If such a description is correct at the relevant scales, the absolute one-way causal barrier presupposed by the present framework is not a feature of the fundamental geometry, only of the semiclassical effective description. The incompleteness theorems remain valid for the semiclassical regime in which they are formulated; the fuzzball proposal asserts that this regime is not fundamental, and therefore that the incompleteness it exhibits is itself an artifact of the effective description.

Effectively finite-dimensional interior Hilbert spaces. Several quantum-gravity proposals—ranging from holographic entropy-bound considerations to discrete approaches such as loop quantum gravity and certain finite-dimensional Hilbert-space conjectures—suggest that the physically admissible interior state space is effectively finite (or at least bounded) at the fundamental level. Theorem 4.10 establishes that within the semiclassical framework the indistinguishability classes are continuum-sized; if the fundamental theory replaces this by a finite or bounded set, the cardinality argument underlying that theorem ceases to apply. The Master Theorem (§4.12) is correspondingly conditional on the third structural ingredient identified in §9.4.

Synthesis. The three programs above are not directly comparable as physical theories, but they share a common formal property: each modifies one of the structural ingredients of the Master Theorem. In this sense, the incompleteness theorems do not adjudicate among quantum-gravity proposals; they characterize the structural cost of restoring operational completeness. A program whose modifications can be identified at the level of these ingredients can be examined for internal consistency; a program whose claimed completeness does not correspond to any such modification is, on the present analysis, in tension with the theorems.

7.4 Philosophical Consequences (Conservative Formulation)

(i) Observer-relative ontology with objective grounding. The dependence of completeness on O does not entail subjectivism: it is determined by the invariant causal domain $J^-(O)$ and by geometric barriers.

(ii) A geometry–logic linkage. The equivalence "geometric indistinguishability $\leftrightarrow$ theoretical underivability" (Theorem 4.13) provides a precise sense in which causal structure fixes logical strength for observer-local theories.

(iii) Limits of "view-from-nowhere" descriptions. Any attempt to demand a single operationally complete description for all observers simultaneously implicitly assumes global causal access. Horizons obstruct this assumption in an ontic (geometric) manner.

7.5 Theoretical Research Directions

(1) Quantitative measures of incompleteness. Define an information-theoretic functional I_O that measures the size/entropy of the equivalence class $[s]_O$ under exterior restriction (e.g., via maximal-entropy completions consistent with exterior data). The aim is a structural invariant of the underdetermination, not an experimentally accessed quantity.

(2) Strengthening the bridge beyond existence. Replace the existence-form Bridge Theorem with robust genericity statements (open dense sets of initial data leading to underivability) under physically standard assumptions. This would strengthen the master theorem from a possibility result to a generic structural feature.

(3) Precise relation to island formulas. Identify the minimal extension of Φ_{ext} needed so that the exterior description becomes effectively injective in the sense relevant to Page-curve computations, and classify which ingredient of Theorem 6.10 is thereby violated. This connects the structural framework here to the technical machinery of replica-wormhole computations.

(4) Cosmological horizons. Apply the same equivalence to de Sitter causal diamonds and quantify the resulting observer-relative underdetermination of global cosmological microstates. The analogous bridge construction for de Sitter requires a separate treatment of admissibility classes, since the asymptotic structure differs essentially from the asymptotically flat case.

(5) Turing-type undecidability via QFT computability. As noted in Remark 8.6 (Section 8.7), the Lawvere fixed-point framework also encompasses Turing undecidability of the halting problem. Whether the causal-geometric mechanism can instantiate Turing-type undecidability—and not merely Gödel-type incompleteness—is a substantive open question. A positive answer would require establishing Turing-completeness of QFT on Schwarzschild backgrounds via a suitable interior encoding of computational steps; the encoding apparatus of §2.4 is potentially adequate for this purpose, but the precise correspondence between physical state evolution and Turing-machine execution remains to be established.

7.6 Comparison with Other Undecidability Results in Physics

The present framework occupies a distinctive position within the broader literature on undecidability and incompleteness in physical theories. It is instructive to situate it explicitly relative to other established results, in order to clarify the precise nature of the mechanism identified here.

Open theoretical questions. Among the directions outlined in §7.5, several remain genuinely open at the level of theoretical structure: whether the bridge theorem admits a generic (open-dense) strengthening; whether the framework can be extended to instantiate Turing-type undecidability; and whether categorical refinements of the Lawvere identification reveal a deeper invariant beyond the specific instances of Gödel and physical diagonalization. These are theoretical questions whose resolution does not depend on empirical input.

Comparative landscape.

Work	Framework	Mechanism	Type of claim
Geroch & Hartle [19]	General relativity	Global computability limits on initial-value problem	Algorithmic / model-theoretic
Cubitt–Pérez-García–Wolf [20]	Many-body / quantum information	Undecidability of the spectral gap	Algorithmic
This work	Semiclassical gravity	Causal-geometric incompleteness via Lawvere fixed-point	Categorical / structural

Key distinction. Many "undecidability in physics" results concern algorithmic or computational undecidability of model properties: whether a given Hamiltonian has a spectral gap, whether the long-time behaviour of a dynamical system is decidable from initial data, and so forth. These are statements about the computational complexity of inferring properties of mathematical models from their defining data. The present work isolates a structurally different mechanism: observer-relative *underivability* enforced by the causal geometry of spacetime in the semiclassical regime, formalized as a categorical instance of Lawvere's fixed-point theorem. The obstruction is not algorithmic intractability but a geometric-logical identification: the same diagonal pattern that makes Gödel's sentence undecidable in arithmetic makes a corresponding interior-state description underivable in Φ_{ext}. The mechanism is therefore neither computational nor empirical, but categorical: it is fixed by the structural relationship between causal restriction and arithmetic representability, and would persist under any change of encoding consistent with that relationship.

8 Ontological Interpretation

This section clarifies in what sense the incompleteness established in Sections 2–6 is geometric (causal-structural) rather than epistemic, and how the "self-referential" construction should be understood ontologically. The goal is not to introduce new technical claims, but to prevent category mistakes: (i) conflating causal underdetermination with practical ignorance, and (ii) reading the triadic correspondence as a merely metaphorical analogy.

8.1 Geometric versus Epistemic Incompleteness

Epistemic incompleteness. By "epistemic" we mean limitations that arise from the contingencies of observation or computation: finite measurement resolution, technological constraints, quantum measurement disturbance, finite computational resources, or chaotic sensitivity. Typical cases include (i) uncertainty tradeoffs for noncommuting observables, (ii) finite predictability horizons in chaotic dynamics, and (iii) intractability barriers in complexity theory.

Geometric (causal-structural) incompleteness. By "geometric" we mean limitations enforced by the causal structure of spacetime relative to a domain D of accessible observables. In the present framework this is exactly causal-geometric incompleteness in the sense of Definition 2.20: there exists a physically realizable state s^* localized in a region R with $R \cap J^-(D) = \varnothing$ such that Φ_D cannot derive a complete description $\mathrm{Desc}(\Phi, s^*)$ (and cannot derive its negation), with the underivability arising from the causal barrier rather than from quantum indeterminacy, gauge redundancy, or complexity.

Key distinction. Even idealized observers with arbitrarily powerful computation and arbitrarily precise exterior measurements cannot overcome a causal exclusion of the relevant degrees of freedom from $J^-(D)$. The obstruction is not "lack of data" but "structural non-injectivity" of the map (global states) $\to$ (exterior data), as quantified by the nontrivial equivalence classes $[s]_D$ of observationally indistinguishable states (Theorem 4.10).

Property	Epistemic	Geometric (causal-structural)
Nature	Limitation of acquisition/processing	Limitation enforced by causal domain
Typical cause	Technology, noise, resources, disturbance	Horizon / causal barrier ($R \cap J^-(D) = \varnothing$)
Resolution	Better access/technology/algorithms	Change the causal-access structure
Observer-dep.	Often contingent	Objective, determined by $J^-(O)$
Formal signature	Hardness/intractability, uncertainty	Non-unique global completion consistent with Obs_D

Theorem 8.1 (Ontological status of causal barriers—within classical GR). *Let (M, g) be a spacetime admitting future null infinity $\mathscr{I}^+$ and define the event horizon $\mathcal{H} = \partial J^-(\mathscr{I}^+)$. Then:*

(1) *$\mathcal{H}$ is diffeomorphism-invariant (a geometric subset of M).*

(2) *Its definition is theory-independent in the sense that it uses only the causal structure of (M, g), not quantum-field details.*

(3) *Its operational consequences are observer-/domain-relative: incompleteness depends on the causal domain $J^-(O)$ (Theorem 4.4).*

Proof sketch. (1) follows because $J^-(\mathscr{I}^+)$ and its boundary are defined purely in terms of the causal relation on (M, g), which is diffeomorphism-invariant. (2) is immediate from the definition. (3) holds because "what can be derived" in Φ_O is restricted to Obs_O, and Obs_O is fixed by $J^-(O)$; changing O changes $J^-(O)$ and hence the induced indistinguishability structure. □

Remark 8.2 (Global versus local). The event horizon is a global object: determining whether a given point lies on $\mathcal{H}$ is not a locally decidable operation from finite-duration measurements. This does not weaken its ontological status as a geometric subset of (M, g); it clarifies that "existence" is a property of the global spacetime, not a locally detectable field.

Structural versus epistemic (precise sense used here). The paper's claim is structural rather than epistemic in the following technical sense: the exterior-restricted theory Φ_{ext} lacks the formal resources to select a unique global description among the physically realizable completions consistent with $\mathrm{Obs}_{\text{ext}}$, because the relevant degrees of freedom lie outside its causal domain. This is logically analogous to underivability in formal systems: enlarging measurement power does not enlarge Φ_{ext}'s domain of derivations if the language and axioms are restricted to the exterior algebra.

8.2 Self-Reference in Physical Systems

Classical (Gödelian) self-reference. In arithmetic, self-reference is syntactic: a sentence G is constructed such that it "talks about" its own Gödel code via diagonalization.

Physical (geometric/operational) self-reference. In this work, "self-reference" is realized by combining: (i) an encoding map γ from (gauge-invariant) physical state data to $\mathbb{N}$ (Section 2.4), (ii) an arithmetic interpretation sufficient for diagonalization (Section 2.6), and (iii) a causal barrier preventing exterior access to the encoded degrees of freedom (Section 3).

The state s^* is "self-referential" in the following operational sense: it is chosen so that its code $\gamma(s^*)$ coincides with the code of a statement asserting non-observability/underivability from the exterior. The horizon supplies the physical analogue of the "unprovability block": it prevents any exterior-local procedure from deciding the interior-encoded predicate.

Crucial clarification (structural, not dynamical). The construction is existential and structural. Nothing requires that generic collapse dynamics produce s^*; the claim is that the semiclassical state space and the encoding capacity admitted by the horizon allow such fixed points to exist while remaining exterior-indistinguishable.

8.3 Structural Realism and Incompleteness (Modest Form)

Structural realism holds that what theories capture most robustly are relational structures rather than intrinsic "things-in-themselves." In the present context, the robust structure is

$$\text{causal accessibility} \rightarrow \text{observational equivalence classes} \rightarrow \text{derivability limits}.$$

formalized by Theorems 4.12–4.13.

On this view, the ontologically salient content is not a particular microstate label, but the invariant structure that (a) partitions state space into equivalence classes relative to Obs_D and (b) constrains what any observer-local theory can uniquely determine. The result supports a moderate structural realism: causal geometry is objective, while the portion of structure accessible to a given observer is fixed by the observer's causal domain.

8.4 Observer-Relative Describability without Subjectivism

A central risk of misinterpretation is to read "observer-relative" as "subjective." In relativity theory, many quantities are observer-dependent without being arbitrary; they are determined by invariant structures (metric, causal cones, four-momentum).

Here the relevant invariant is the causal domain $J^-(O)$ (or, more generally, the domain D defining Obs_D). External and infalling observers differ because their causal access differs:

O_{ext}: $R_{\mathrm{int}} \cap J^-(O_{\mathrm{ext}}) = \varnothing$, hence Φ_{ext} is incomplete (Theorem 4.4).

O_{inf}: prior to the singularity, $J^-(O_{\mathrm{inf}})$ intersects portions of the interior, hence the relevant observer-local theory may become more informative.

Accordingly, "observer-relative ontology" is best stated as *observer-relative describability*: what can be uniquely described by Φ_O depends on O's causal position, but this dependence is fixed by objective geometry.

8.5 Summary

Within the scope of this paper (exterior-restricted semiclassical gravity in spacetimes with horizons), the incompleteness results admit the following structural reading:

(1) Geometric grounding: the obstruction is enforced by causal structure, not by practical limits.

(2) Structural non-injectivity: exterior data admit multiple physically realizable global completions, with the indistinguishability class having continuum cardinality (Theorem 3.6).

(3) Physical self-reference (operational): diagonalization is realized by interior encoding combined with causal inaccessibility.

(4) Observer-relativity with objective grounding: describability varies with $J^-(O)$, not with belief, technology, or interpretive choice.

(5) No "view from nowhere" at the semiclassical exterior level: completeness in the operational sense requires either access to the full causal domain, or a framework that modifies one of the structural ingredients identified by the Master Theorem (§9.4).

8.6 The Triadic Correspondence as an Ontological Constraint

Section 2.8 established a triadic correspondence between *Geometry* (causal structure of the DOC), *Physics* (exterior observable algebras and state restriction), and *Logic* (formal systems obtained via Gödel encoding of the restricted theory), implemented by functors F, G, H under the scope convention adopted there ($\mathbf{Cat}_L$ restricted to the essential image of $G \circ F$).

Ontological reading (restricted). Under these conventions, "inaccessibility," "indistinguishability," and "underivability" are not three unrelated phenomena but three presentations of the same structural constraint:

$$\text{Geometric Inaccessibility} \iff \text{Physical Indistinguishability} \iff \text{Logical Underivability}.$$

as formalized by Theorem 2.17 and the Bridge Theorem (Theorem 2.12) together with the equivalence results in Section 4.

No-privilege thesis (limited). The correspondence does not imply that geometry, physics, and logic are metaphysically identical; it implies that, for the class of systems considered and for the observer-local restrictions defined, completeness/incompleteness is constrained by an invariant structure that can be expressed equivalently in geometric, physical, or logical terms. In particular, any attempted "completion" that restores unique describability must, in effect, modify at least one vertex of the triad (e.g., by changing causal locality, enlarging the accessible algebra, or altering the encoding/interpretation conditions), as anticipated in Sections 5–7.

8.7 Lawvere's Fixed-Point Theorem and the Categorical Equivalence of Diagonal Arguments

The preceding sections established a structural parallel between Gödelian incompleteness and causal-geometric incompleteness. We now show that both are instances of a single abstract categorical theorem, thereby establishing a categorical equivalence at the level of fixed-point architecture: the two diagonal constructions are isomorphic instances of Lawvere's theorem under an explicit, computable code translation.

The abstract theorem

Lawvere's fixed-point theorem [51] provides a unified categorical framework for all classical diagonal arguments (Cantor, Gödel, Turing, Tarski). We state a version sufficient for our purposes.

Theorem 8.3 (Lawvere 1969). *Let $\mathcal{C}$ be a category with a terminal object. Suppose there exist objects A and B and a* point-surjective *morphism $\hat{\phi}\colon A\times A \to B$ (i.e., for every morphism $g\colon A \to B$, there exists a point $a\colon 1 \to A$ such that $g = \hat{\phi}\circ(\mathrm{id}_A \times a)$ as maps $A \to B$). Then every endomorphism $f\colon B \to B$ has a fixed point: there exists a point $p\colon 1 \to B$ such that $f \circ p = p$.*

The contrapositive yields all diagonal impossibility results: if there exists a *fixed-point-free* endomorphism $f\colon B \to B$ (e.g., negation), then no point-surjective $\hat{\phi}\colon A \times A \to B$ can exist. The classical corollaries are:

Cantor: $A = \mathbb{N}$, $B = \{0,1\}$, $f = \text{negation} \Rightarrow$ no surjection $\mathbb{N} \to 2^{\mathbb{N}}$.

Gödel: $A = \text{sentences}$, $B = \{\text{provable}, \text{unprovable}\}$, $f = \text{negation} \Rightarrow$ existence of self-referential undecidable sentence.

Turing: $A = \text{programs}$, $B = \{\text{halts}, \text{loops}\}$, $f = \text{negation} \Rightarrow$ halting problem undecidable.

Application to causal-geometric incompleteness

We now show that the physical diagonal construction of Section 4 is an additional instance of Lawvere's theorem, in the following precise sense.

Proposition 8.4 (Physical incompleteness as Lawvere fixed point). *The diagonal construction of §4.1–4.3 instantiates Lawvere's fixed-point theorem (Theorem 8.3) in the category* **Set** *with:*

- *$A := \mathbb{N}$ (the set of Gödel codes at resolution N, identified with finite-resolution physical state descriptions via γ_N),*
- *$B := \{0,1\}$ (representing $\{$derivable in Φ_{D_O}, not derivable in $\Phi_{D_O}\}$),*
- *$\hat{\phi}\colon \mathbb{N}\times\mathbb{N} \to \{0,1\}$ defined by*

$$\hat{\phi}(n,m) := \begin{cases} 1 & \text{if } \Phi_{D_O} \vdash \theta_n(m)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

  where θ_n is the formula with Gödel number n,
- *$f := \text{negation}\colon \{0,1\} \to \{0,1\}$, $f(b) = 1 - b$.*

Since f is fixed-point-free, Lawvere's theorem (contrapositive) implies that $\hat{\phi}$ is not *point-surjective. Concretely, the diagonal map $d\colon \mathbb{N} \to \{0,1\}$ defined by $d(n) := f(\hat{\phi}(n,n)) = 1 - \hat{\phi}(n,n)$ is not in the range of $\hat{\phi}(\cdot,a)$ for any a. This is precisely the function "$n \mapsto$ 'Φ_{D_O} does not prove $\theta_n(n)$'"—and the fixed-point sentence G^* of §4.1 is the sentence with $n^* = \ulcorner G^* \urcorner$ satisfying $d(n^*) = 1$, i.e., G^* asserts its own non-derivability.*

The physical content *enters through the identification $A \ni n \longleftrightarrow s_n \in \mathcal{S}_{\mathrm{adm}}$ (admissible interior states via the encoding γ_N), and through the fact that the domain restriction D_O (enforced by the causal barrier) is what prevents $\hat{\phi}$ from being point-surjective: interior-supported codes are not accessible to Φ_{D_O}-derivations grounded in* Obs_{D_O}.

Proposition 8.5 (Isomorphism of specific Lawvere instances). *Let $(N, \{0,1\}, \hat{\phi}_G, f)$ be the standard Gödel Lawvere instance for a consistent recursively axiomatizable theory F extending Q, and let $(N, \{0,1\}, \hat{\phi}_P, f)$ be the physical Lawvere instance of Proposition 8.4 for Φ_{D_O}. Then the two instances are isomorphic as specific Lawvere fixed-point instances in the following sense: there exists a recursively invertible bijection $\sigma : N \to N$ (the code translation induced by γ_N composed with the syntactic Gödel numbering) such that*

(i) ***Evaluation maps correspond:*** *$\hat{\phi}_P(\sigma(n), \sigma(m)) = \hat{\phi}_G(n,m)$ for all $n, m \in \mathbb{N}$. This isomorphism operates at the level of the specific diagonal construction, not at the level of a global equivalence of categories; the latter is neither claimed nor required for the incompleteness results.*

(ii) ***Negations coincide:*** *$f_P = f_G = (b \mapsto 1 - b)$.*

(iii) ***Diagonal maps correspond:*** *$d_P \circ \sigma = d_G$, where $d_G(n) := 1 - \hat{\phi}_G(n,n)$ and $d_P(n) := 1 - \hat{\phi}_P(n,n)$.*

Consequently, the fixed-point sentence G^* and its Gödel-theoretic counterpart G_F are related by $n^* = \sigma(\ulcorner G_F \urcorner)$, and the undecidability of G^* in Φ_{D_O} follows from the undecidability of G_F in F via this isomorphism.

Proof. The functor $G \circ F$ (Section 2.8) maps the exterior algebra $\mathfrak{A}(D_O)$ to a formal system $\mathcal{L}_{A,N}$ that, by construction, is bi-interpretable with the arithmetized version of Φ_{D_O} at resolution N. The code translation σ is the composition of (a) the syntactic Gödel numbering $\ulcorner\cdot\urcorner$ of F-sentences and (b) the physical encoding γ_N applied to the interior realization of those codes. By Proposition G.3 (recursive equivalence of acceptable Gödel numberings), σ is total computable with a total computable inverse.

For (i): by the definition of $\hat{\phi}_P$ and $\hat{\phi}_G$, both evaluate "does the respective theory prove $\theta_n(m)$?" The bi-interpretability ensures that $F \vdash \theta_n(m)$ if and only if $\Phi_{D_O} \vdash \theta_{\sigma(n)}(\sigma(m))$, which is precisely $\hat{\phi}_P(\sigma(n), \sigma(m)) = \hat{\phi}_G(n, m)$.

For (ii): both use Boolean negation on $\{0, 1\}$.

For (iii): $d_P(\sigma(n)) = 1 - \hat{\phi}_P(\sigma(n), \sigma(n)) = 1 - \hat{\phi}_G(n, n) = d_G(n)$.

□

Remark 8.6 (What the isomorphism achieves). (i) Isomorphism of specific Lawvere instances. Gödel's syntactic self-reference and our geometric self-reference are **isomorphic as specific Lawvere fixed-point instances**: the evaluation maps $\hat{\phi}_P(\sigma(n), \sigma(m)) = \hat{\phi}_G(n, m)$, the diagonal maps $d_P \circ \sigma = d_G$, and the fixed points are connected by an explicit, computable, invertible translation σ mediated by the triadic correspondence. This is a precise isomorphism at the level of the diagonal construction (preservation of the exact combinatorial structure of the Lawvere diagram), not a global equivalence of categories. The latter is neither claimed nor required for the incompleteness results.

(ii) Inevitability. Lawvere's theorem shows that the diagonal obstruction is not an artifact of a particular encoding but is forced by the categorical structure whenever the relevant evaluation map exists and the codomain admits a fixed-point-free endomorphism. This explains why our construction is robust under changes of Gödel numbering (Proposition G.3).

(iii) Scope. The Lawvere framework also encompasses the Turing undecidability of the halting problem (same abstract pattern, different substrate). This suggests a broader research program: investigating whether the causal-geometric mechanism can instantiate Turing-type undecidability (not merely Gödel-type), which would require establishing Turing-completeness of QFT on Schwarzschild backgrounds — a question we leave for future work (§7.5).

9 Physical Interpretation and Scope of the Incompleteness Results

The mathematical content of this paper consists of three theorems and their unification in the Master Incompleteness Theorem (Theorem 4.14), establishing a structural identity between causal-geometric inaccessibility and Gödel-type underivability via Lawvere's fixed-point theorem. The present section clarifies how these results bear on physical theory: what they entail, what they do not entail, and how they relate to existing developments in semiclassical gravity and algebraic quantum field theory. The discussion is deliberately interpretive rather than predictive: the framework asserts a structural impossibility, not a positive empirical signature, and we are careful throughout to preserve this distinction.

9.1 Status of the Claims

We restate the distinction introduced in §1.7 in sharper form.

Theorems as conditional implications. Theorems 2.12, 4.13, 6.10, and 4.14 are conditional mathematical statements: *if* a theory Φ admits an interpretation of Robinson Arithmetic Q, operates from a causally limited observational domain D_O, and describes spacetime with absolute causal barriers, *then* Φ exhibits the specified incompleteness. The validity of these implications is independent of any empirical fact, exactly as the validity of Gödel's incompleteness theorems is independent of whether arithmetic is ever applied to anything.

Application to semiclassical gravity. The applicability of the antecedent to Φ_{semi} rests on three independently supported claims: (i) general relativity, repeatedly confirmed by gravitational-wave observations and by precision tests in the strong-field regime; (ii) the existence of event horizons, supported both by the singularity theorems of Penrose [48] and by direct imaging of supermassive compact objects; and (iii) the validity of semiclassical QFT in the regime relevant to the bridge construction. None of these is an empirical prediction of the present work; they are background assumptions whose status is independent of the incompleteness theorems.

What the framework does not assert. The framework makes no claim about what astrophysical, laboratory, or analog measurements *would observe* in any specific apparatus. It asserts a structural fact about the inverse problem any exterior-restricted semiclassical theory must face: the map from global states to exterior data is non-injective in a precise, categorically forced sense (cf. Theorem 4.10 and Proposition 8.4).

9.2 Relation to the Existing Literature

The incompleteness theorems clarify, rather than contest, several established lines of work in semiclassical gravity and algebraic QFT.

Page-curve and island constructions. The replica-wormhole and quantum-extremal-surface results of Penington [43], Engelhardt and Wall [44], and Almheiri *et al.* [45] reproduce a unitary Page curve by extending the semiclassical computation with nonlocal contributions, including entanglement wedges and replica geometries. In the language of the present framework, these constructions exemplify the kind of extension required to restore exterior injectivity: they go beyond the strictly exterior-restricted theory Φ_{ext} as defined in §2–4, precisely because Φ_{ext} is provably incomplete in the sense of Theorem 4.14. Our theorems thus provide a structural rationale for why such extensions are necessary, not merely convenient: any unitarizing completion of Φ_{ext} must violate at least one of the three structural ingredients identified in §9.4 below.

Algebraic quantum field theory. The split property of Doplicher and Longo [47] and the modular-theoretic framework of Bisognano and Wichmann [50] establish algebraic-independence properties of local algebras associated to causally disjoint regions. The geometric-indistinguishability content of Theorem 4.13 can be read as a categorical strengthening of these properties: not merely that exterior and interior algebras are independent, but that this independence *forces* underivability of an entire class of interior facts within any exterior-local axiomatization that interprets Q.

Quantum energy inequalities. Fewster's quantum energy inequalities [49] replace the classical pointwise energy conditions in the Hadamard-admissibility requirements employed throughout this paper (cf. §3.6). The compatibility of Theorem 6.10 with these inequalities is significant: it shows that the incompleteness mechanism does not depend on energy-condition violations and persists in the class of states physically admissible in curved-spacetime QFT.

9.3 Operational Consequences

The theorems entail an inverse-problem statement that admits operational formulation independently of any specific experimental program.

Inverse-problem non-uniqueness. Let $\mathcal{S}_{\mathrm{global}}$ denote the space of global semiclassical states and $\rho_{\mathrm{ext}} \colon \mathcal{S}_{\mathrm{global}} \to \mathcal{S}_{\mathrm{ext}}$ the restriction map to exterior data. Theorem 4.10 entails that ρ_{ext} has fibers of cardinality at least $2^{\aleph_0}$ over generic exterior data when an effective horizon is present. This is a structural property of the map, not a measurement-precision artifact: it persists in the limit of arbitrarily fine exterior measurement and arbitrarily large data sets.

Domain-relativity of completeness. The notion of completeness is indexed by a causal domain. A description that is incomplete with respect to D_O may be complete with respect to a strictly larger domain $D_{O'} \supsetneq D_O$ (e.g., for an infalling observer whose causal past intersects the interior). This is not a relativization to subjective standpoint, but to the objective causal structure of spacetime (cf. §4.10). The framework therefore identifies completeness as a relational property between a theory and a domain, not as an absolute property of theories.

9.4 Conditions for Restoration of Operational Completeness

The Master Theorem (Theorem 4.14) identifies three structural ingredients whose joint presence forces incompleteness: (1) arithmetic representability of the interior algebra, (2) causal-domain restriction enforced by the horizon, and (3) sufficient interior state-space cardinality. Any extension of Φ_{ext} that restores operational completeness must violate at least one of these. We discuss each escape route in turn.

Modification of causal locality. Holographic duality, replica-wormhole mechanisms, and related nonlocal constructions violate (2): they grant the exterior description effective access to information that, in the strictly local sense of Φ_{ext}, lies behind the horizon. The Page-curve results discussed in §9.2 are instances of this strategy. The framework predicts, in this sense, that any successful unitarizing completion must reorganize the locality structure of Φ_{ext} in an essential way.

Effective discretization of the interior state space. If a putative theory of quantum gravity renders the interior state space effectively finite (rather than continuum-sized at the relevant level of description), ingredient (3) is weakened. This is a structural prediction about what any unitarizing completion must accomplish: the continuum of indistinguishability classes documented in Theorem 4.10 must collapse to a discrete (and ideally bounded) set at the fundamental level. This route is consistent with motivations underlying holographic entropy bounds and finite-dimensional Hilbert-space proposals.

Failure of arithmetic representability. If the encoding map $\iota \colon \mathrm{Sent}(F) \to \mathcal{S}_{\mathrm{int}}$ used in the bridge fails—e.g., because the interior algebra cannot, in any extension, represent Q—ingredient (1) is violated. We regard this as the least plausible escape route, since arithmetic representability is generically present whenever the interior admits a sector with arbitrary occupation numbers, and the encoding constructed in §2.4 is robust under variations of the gauge-invariant numbering scheme (Proposition G.3).

9.5 Falsification Within Scope

Although the theorems are mathematical rather than phenomenological, their conditional structure admits precise falsification criteria at the level of the antecedent.

Falsification of the bridge construction. If a future development establishes that semiclassical gravity, formulated with an exterior-restricted observable algebra, admits no encoding map ι satisfying clause (B) of Theorem 2.12—that is, that the interior cannot, even in principle, support arithmetic representability—then the application of the framework to Φ_{semi} fails. The mathematical theorems remain valid as conditionals; only their physical applicability is withdrawn.

Falsification of the geometric premise. If absolute causal barriers turn out to be artifacts of the semiclassical approximation rather than features of any successor theory—e.g., if quantum gravity smooths out horizons in a way that eliminates the strict J^-/J^+ separation at the relevant length scale—then ingredient (2) of the Master Theorem becomes inapplicable in that successor regime. This would not refute the theorems but would restrict their domain to the effective regime in which causal barriers exist as a robust geometric feature.

What would not falsify the framework. The non-observation of any specific phenomenological signature (whether in analog-gravity platforms, gravitational-wave inference, or relativistic quantum information) does not bear on the validity of the theorems. The framework asserts a structural impossibility forced by the joint presence of the three ingredients of Theorem 4.14; absence of a positive observational signature is fully consistent with this structural claim, and indeed is what one would expect if the inverse-problem non-uniqueness is exactly as the theorems predict.

In summary, the present results are best understood as a *constraint theorem*: any candidate completion of semiclassical gravity must, on pain of contradiction with Theorem 4.14, either modify causal locality, render the interior state space effectively finite, or break arithmetic representability. The framework does not predict what such a completion will look like; it specifies what it cannot avoid.

10 Conclusions

We have established that semiclassical gravity admits a form of theoretical incompleteness that is enforced by the causal geometry of spacetime. The principal results—geometric underdetermination, Gödel-type undecidability, and their categorical equivalence via Lawvere's fixed-point theorem—are unified in the Master Incompleteness Theorem (Theorem 4.14), with explicit hypotheses (recursive axiomatizability with arithmetic interpretation, split property, consistency, Σ_1-soundness) and logically independent conclusions. When a theory is restricted to a causal domain D (e.g., the exterior DOC of a black hole), the presence of a causal barrier implies that physically realizable states localized in the causally inaccessible complement can fail to admit complete descriptions derivable within the restricted theory Φ_D. Concretely, horizons generate nontrivial observational equivalence classes of global states relative to the accessible observable algebra, and this non-injectivity entails underivability of unique state descriptions in Φ_D.

The significance of the result is not that Gödel's incompleteness theorems are "imported" into physics in a literal sense. Rather, we have shown that causal structure can enforce a Gödelian pattern—self-reference together with undecidability—inside otherwise standard, consistent, recursively axiomatizable semiclassical frameworks. In this setting, incompleteness is structural and geometric, not epistemic: it persists even under idealized limits of measurement precision and computational power, because the relevant degrees of freedom lie outside the causal past of the observer/domain.

Black holes provide the canonical and technically transparent realization of this mechanism, since event horizons are absolute one-way causal barriers and the Bekenstein–Hawking entropy provides ample encoding capacity for the interior microstructure required by the diagonal construction. However, the mechanism is not restricted to Schwarzschild black holes: by the minimal incompleteness criterion (Theorem 6.10), any physical situation satisfying the three structural ingredients—existence of a causal barrier, nontrivial inaccessible state space, and split/independence of the inaccessible sector—admits the same incompleteness pattern. Section 6 establishes this for rotating and charged black holes, higher-dimensional Schwarzschild–Tangherlini geometries, cosmological (de Sitter) horizons, Rindler horizons for accelerated observers, and analog horizon systems.

Constraint-theoretic significance. The framework developed here is best understood as a *constraint theorem* on candidate completions of semiclassical gravity (§9.5). Any extension that restores operational completeness must, on pain of contradiction with the Master Theorem, modify at least one of three structural ingredients (§9.4): the strict causal locality of Φ_{ext}, the cardinality of the admissible interior state space, or the arithmetic representability that supports the diagonal construction. Each of the principal lines in the quantum-gravity literature—holographic dualities and replica-wormhole / quantum-extremal-surface constructions, fuzzball-type microstate geometries, and finite-dimensional Hilbert-space proposals—can be located, on the present analysis, by which structural ingredient it modifies. The framework does not adjudicate among these proposals; it specifies the structural cost that any of them must pay.

Accordingly, the appropriate reading is not that "nature is unknowable," but that observer-local theories respecting classical causal barriers cannot, in principle, select a unique global completion among all physically realizable states consistent with their accessible observables. This reframes the black hole information problem (§7.1): the mixedness of the exterior description is the operational signature of an intrinsic structural non-injectivity of the restriction map, and any unitarizing completion must transcend the semiclassical causal framework in one of the three precise senses identified by the Master Theorem.

Open theoretical directions. Several theoretical questions remain open within the present framework (§7.5): the strengthening of the Bridge Theorem from existence to genericity (open dense sets of initial data leading to underivability); the precise identification of which structural ingredient is modified in each replica-wormhole / island construction; the extension of the framework to de Sitter causal diamonds with appropriate adaptation of the admissibility class; and the question of whether the causal-geometric mechanism can instantiate Turing-type undecidability via QFT computability on Schwarzschild backgrounds (Remark 8.6). These are theoretical questions whose resolution does not depend on empirical input.

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A Explicit Calculations

A.1 Schwarzschild Metric in Different Coordinates

Schwarzschild coordinates (t, r, θ, φ):

$$ds^2 = -\Big(1 - \frac{r_\mathrm{s}}{r}\Big)c^2\,dt^2 + \Big(1 - \frac{r_\mathrm{s}}{r}\Big)^{-1}dr^2 + r^2\,d\Omega^2\,,$$

where $r_\mathrm{s} = 2GM/c^2$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$.

Ingoing Eddington–Finkelstein coordinates (v, r, θ, φ):

$$ds^2 = -\Big(1 - \frac{r_\mathrm{s}}{r}\Big)c^2\,dv^2 + 2c\,dv\,dr + r^2\,d\Omega^2\,,$$

where $v = t + r_*/c$ and $r_* = r + r_\mathrm{s}\ln|r/r_\mathrm{s} - 1|$.

Kruskal–Szekeres coordinates (T, X, θ, φ):

$$ds^2 = \frac{4r_\mathrm{s}^3}{r}\,e^{-r/r_\mathrm{s}}\big(-c^2\,dT^2 + dX^2\big) + r^2\,d\Omega^2\,,$$

with implicit relation $T^2 - X^2 = (1 - r/r_\mathrm{s})\,e^{r/r_\mathrm{s}}$.

All three coordinate systems describe the same maximal analytic extension of the Schwarzschild geometry. In every system, the event horizon is the null hypersurface $\mathcal{H} = \partial J^-(\mathscr{I}^+)$, a diffeomorphism-invariant definition independent of coordinate choices.

A.2 Bekenstein–Hawking Entropy (Schwarzschild)

For a Schwarzschild black hole of mass M:

Horizon area: $A = 4\pi r_\mathrm{s}^2 = 16\pi G^2M^2/c^4$.

Bekenstein–Hawking entropy:

$$S_\mathrm{BH} = \frac{k_\mathrm{B}\,c^3\,A}{4\,G\,\hbar} = \frac{4\pi\,k_\mathrm{B}\,G\,M^2}{\hbar\,c}\,.$$

For $M = M_\odot \approx 1.99 \times 10^{30}$ kg: $S_\mathrm{BH}/k_\mathrm{B} \approx 1.05 \times 10^{77}$.

Number of microstates (order-of-magnitude): $N_\mathrm{max} \sim \exp(S_\mathrm{BH}/k_\mathrm{B})$, equivalently $\log_{10}(N_\mathrm{max}) = (S_\mathrm{BH}/k_\mathrm{B})/\ln 10 \approx 4.56 \times 10^{76}$.

A.3 Gödel Numbering: Encoding Capacity (Order-of-Magnitude)

A standard Gödel encoding for arithmetic requires a finite symbol alphabet, typically $O(10\text{–}10^2)$ primitive symbols. Gödel codes of syntactic objects grow at most exponentially in the length of the object, and in practice the Gödel numbers needed to represent concrete provability predicates are astronomically smaller than $\exp(S_\mathrm{BH}/k_\mathrm{B})$ for astrophysical black holes. For a solar-mass black hole, $S_\mathrm{BH}/k_\mathrm{B} \sim 10^{77}$ bits vastly exceeds any fixed arithmetization used in Gödel-type diagonal constructions.

B Supporting Lemmas for the Bridge Theorem

This appendix provides technical statements supporting Theorem 2.12 (Bridge Theorem).

Lemma B.1 (Exterior restriction and split factorization)**.** *Let (M, g) be Schwarzschild spacetime with event horizon at $r = r_\mathrm{s}$, and fix a buffer scale $\delta \gg \ell_\mathrm{P}$. Define the buffered exterior and interior regions $R_\mathrm{ext} := \{r > r_\mathrm{s} + \delta\}$, $R_\mathrm{int} := \{r < r_\mathrm{s} - \delta\}$. Let $\mathfrak{A}(R)$ denote the (von Neumann) algebra of observables localized in region R for a QFT on (M, g), and write $\mathfrak{A}_\mathrm{ext} := \mathfrak{A}(R_\mathrm{ext})$, $\mathfrak{A}_\mathrm{int} := \mathfrak{A}(R_\mathrm{int})$.*

Assume the split property for the inclusion of local algebras across the collar region between R_int and R_ext (standard in algebraic QFT for Hadamard states, physically justified by $\delta \gg \ell_\mathrm{P}$). Then there exists a Type I factor $\mathcal{N}$ such that the generated algebra admits an (approximate) tensor product structure:

$$\mathfrak{A}_\mathrm{ext} \vee \mathfrak{A}_\mathrm{int} \;\simeq\; \mathfrak{A}_\mathrm{ext} \mathbin{\bar{\otimes}} \mathfrak{A}_\mathrm{int}\,,$$

with corrections suppressed by the collar scale. In particular, for any product state $\omega = \omega_\mathrm{ext} \otimes \omega_\mathrm{int}$ and any $A \in \mathfrak{A}_\mathrm{ext}$, one has $\omega(A) = \omega_\mathrm{ext}(A)$, so distinct interior states can yield identical exterior data.

Proof sketch. By the split property, there exists an intermediate Type I factor implementing a tensor-product decomposition between the buffered algebras. The identity $\omega(A \otimes \mathbf{1}) = \omega_\mathrm{ext}(A)$ then follows immediately. The restriction map $\omega \mapsto \omega|_{\mathfrak{A}_\mathrm{ext}}$ is not injective whenever $\mathfrak{A}_\mathrm{int}$ is nontrivial. □

Remark B.2 (Status of the split property in curved spacetime). The split property for local algebras is rigorously established for free scalar fields in Minkowski spacetime [47, 54]. In curved spacetime, the split property for Hadamard states on globally hyperbolic backgrounds has been proved by Verch [55] for the free scalar field and is expected to hold more broadly under the standard axioms of locally covariant QFT [37, 56]. For interacting fields, the split property remains a conjecture, though it is strongly motivated by the construction of local thermal equilibrium states and by perturbative AQFT [56]. In our framework, (H2) is stated as an explicit hypothesis precisely because a fully rigorous proof in the Schwarzschild interior–exterior setting with interacting matter content is not yet available in the literature. The incompleteness conclusions are conditional on (H2); should the split property fail in some regime, the scope of the result would be correspondingly narrowed.

Lemma B.3 (Distinguished Unruh reference state). *Fix the Schwarzschild geometry and impose the asymptotic boundary condition of no incoming radiation from $\mathscr{I}^-$ appropriate to gravitational collapse. Then there exists a distinguished Hadamard state ω_U (the Unruh state) characterized by: (i) regularity across the future event horizon, (ii) absence of incoming flux from $\mathscr{I}^-$, (iii) Hawking-thermal outgoing flux at $\mathscr{I}^+$ with temperature T_H.*

Within the class of quasifree (Gaussian), Hadamard states satisfying (i)–(ii), ω_U is singled out up to finite-energy exterior excitations.

Lemma B.4 (Closure under $*$-algebra operations). *Let ω_1 and ω_2 be states on the same observable $*$-algebra $\mathfrak{A}(D_O)$. If $\omega_1(A) = \omega_2(A)$ for all $A \in \mathfrak{A}(D_O)$, then $\omega_1(P) = \omega_2(P)$ for every element P in the $*$-algebra generated by $\mathfrak{A}(D_O)$.*

Proof. Polynomials in elements of $\mathfrak{A}(D_O)$ are themselves elements of $\mathfrak{A}(D_O)$ (or its generated $*$-algebra). Since ω_1 and ω_2 coincide on the algebra, they coincide on all such elements. □

Lemma B.5 (Energy admissibility of interior encoding states). *In the semiclassical regime $M \gg m_{\mathrm{P}}$, there exist interior encoding states supported in R_{int} whose renormalized stress-energy expectation $\langle T_{\mu\nu}\rangle_{\mathrm{ren}}$ is physically admissible in the following sense:*

(A) Positivity of total (ADM) energy shift: $E_{\mathrm{int}} \geq 0$.

(B) Smallness (backreaction control): $E_{\mathrm{int}} \ll Mc^2$, *hence $\delta g/g \lesssim E_{\mathrm{int}}/(Mc^2) \ll 1$.*

(C) Local energy boundedness (QFT sense): *the smeared energy density satisfies a quantum energy inequality (QEI) bound [49] with finite $C(f,\gamma)$ (Hadamard renormalization).*

Proof sketch. Hadamard states admit well-defined renormalized stress tensors, and QEIs hold for free fields. Finite-particle excitations have nonnegative total energy and can be localized to R_{int} with arbitrarily small total excitation energy by choosing low occupation numbers. The backreaction estimate follows from the ADM mass shift $\delta M = E_{\mathrm{int}}/c^2$. □

Lemma B.6 (Negligible backreaction bound). *Let a Schwarzschild black hole of mass M support an interior excitation of total energy E_{int}. In linearized semiclassical gravity, $\delta g/g \sim E_{\mathrm{int}}/(Mc^2)$.*

Numerical estimate *($M = M_\odot$): With $r_{\mathrm{s}} \approx 2.95 \times 10^3$ m and $\omega \sim c/r_{\mathrm{s}}$, $\hbar\omega \approx 1.1 \times 10^{-29}$ J, while $Mc^2 \approx 1.8 \times 10^{47}$ J. Hence for one quantum, $\delta g/g \sim O(10^{-76})$, and for n_q quanta, $\delta g/g \sim n_q \times 10^{-76}$.*

Lemma B.7 (Injectivity of gauge-invariant Gödel encoding). *Let $\{\hat{O}_i\}_{i\in\mathbb{N}}$ be a countable separating family of gauge-invariant observables, and let $\varepsilon_i > 0$ be fixed discretization scales. Define the physical equivalence relation $[\psi] \equiv [\psi']$ iff $|\langle\hat{O}_i\rangle_\psi - \langle\hat{O}_i\rangle_{\psi'}| < \varepsilon_i$ for all i. Define $\gamma([\psi])$ by encoding the discretized data via an injective computable pairing (e.g., prime-power coding with unique factorization). Then γ is injective on the quotient space of equivalence classes.*

Proof. If $\gamma([\psi]) = \gamma([\psi'])$, unique decoding implies $\lfloor\langle\hat{O}_i\rangle_\psi/\varepsilon_i\rfloor = \lfloor\langle\hat{O}_i\rangle_{\psi'}/\varepsilon_i\rfloor$ for all i, hence the discretized data coincide, and therefore $[\psi] \equiv [\psi']$. □

Summary of Appendix B. Lemma B.1 supplies the split-property justification for the operational interior/exterior decomposition. Lemma B.3 fixes the exterior reference state. Lemma B.4 formalizes stability of observational equivalence under algebraic constructions. Lemma B.5 ensures physical realizability of interior excitations. Lemma B.6 quantifies backreaction control. Lemma B.7 ensures well-defined injective Gödel encoding.

C Gauge-Invariant Encoding

Problem. Gauge-related states $|\psi\rangle$ and $U_g|\psi\rangle$ ($g \in \mathcal{G}$) are physically equivalent. A naïve Gödel encoding on raw vectors would spuriously distinguish gauge copies.

Solution. Define the encoding on gauge-equivalence classes $[\psi] = \{U_g|\psi\rangle : g \in \mathcal{G}\}$, using only gauge-invariant data.

Construction. *Step 1.* Choose a countable separating set of gauge-invariant observables $\{\hat{O}_i\}_{i\in\mathbb{N}}$. *Step 2.* For a representative $|\psi\rangle$ of the class $[\psi]$, define $r_i = \langle\hat{O}_i\rangle_\psi$, which is independent of the representative. *Step 3.* Discretize to rational data, e.g., $\tilde{r}_i = \lfloor 10^N r_i\rfloor/10^N \in \mathbb{Q}$. *Step 4.* Encode the sequence $\{\tilde{r}_i\}$ by an injective computable map γ, e.g., via prime-power coding: $\gamma([\psi]) = \prod_{i=1}^M p_i^{\mathrm{enc}(\tilde{r}_i)}$.

Gauge invariance. Since each $\hat{O}_i$ is gauge-invariant, $\langle\hat{O}_i\rangle_{U_g\psi} = \langle\hat{O}_i\rangle_\psi$, hence $\gamma([U_g\psi]) = \gamma([\psi])$.

D Representability and the Diagonal Lemma

D.1 The Standard Diagonal Lemma (Classical Statement)

Let T be any consistent, recursively axiomatizable theory interpreting Robinson Arithmetic Q. For any formula $\psi(x)$ with one free variable, there exists a sentence φ such that $T \vdash \varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner)$, where $\ulcorner\varphi\urcorner$ denotes the Gödel number of φ.

D.2 Applicability to Semiclassical Gravity (Scope Statement)

By the arithmetic interpretation established in Appendix E, the semiclassical framework Φ_{semi} supports an interpretation of Robinson Arithmetic. Under the standing assumption that Φ_{semi} admits a recursively enumerable axiom set within its regime of applicability (as specified in §2.2), the diagonal lemma applies to the induced formal system.

D.3 Physical Realization of the Diagonal Fixed Point

Let $\psi(n)$ be the predicate expressing that "the state with code n is not uniquely derivable from observations restricted to the exterior domain D_O." By the diagonal lemma there exists φ^* with $\Phi_O \vdash \varphi^* \leftrightarrow \psi(\ulcorner\varphi^*\urcorner)$. Let $n^* = \ulcorner\varphi^*\urcorner$ and choose an interior state $|s^*\rangle$ such that $\gamma(s^*) = n^*$. This yields the physical fixed point used in Section 4.

D.4 Undecidability (Exterior Restriction)

The exterior-restricted theory Φ_O cannot decide the corresponding completeness/describability statement for s^*:

$$\Phi_O \nvdash \mathrm{Desc}(\Phi, s^*) \qquad \text{and} \qquad \Phi_O \nvdash \neg\mathrm{Desc}(\Phi, s^*)\,,$$

under the hypotheses stated in the main text (consistency/soundness in regime and causal restriction). This is the precise sense in which the diagonal construction produces an undecidable physical state relative to the exterior theory.

E Arithmetic Interpretation in QFT

This appendix records an explicit realization of Robinson Arithmetic Q within multi-mode Fock space, sufficient for the Gödelian diagonal constructions used in the main results.

E.1 Natural Numbers in Fock Space

Let $\mathcal{H}$ be the single-mode Fock space with number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ and eigenbasis $\{|n\rangle\}_{n\in\mathbb{N}}$: $\hat{N}\,|n\rangle = n\,|n\rangle$. Define $0 \mapsto |0\rangle$. Define the successor partial isometry $\hat{V}\,|n\rangle = |n+1\rangle$ for all $n \in \mathbb{N}$ (equivalently, $\hat{a}^\dagger = \hat{V}\sqrt{\hat{N}+1}$). This realizes the basic successor structure required by Q1–Q3.

E.2 Addition as a Deterministic Observable

Work in the two-mode space $\mathcal{H}_A \otimes \mathcal{H}_B$ with number operators $\hat{N}_A$ and $\hat{N}_B$. Define the addition observable $\hat{N}_+ := \hat{N}_A + \hat{N}_B$. On product number states: $\hat{N}_+(|m\rangle_A\,|n\rangle_B) = (m+n)\,|m\rangle_A\,|n\rangle_B$. The sum $m+n$ is obtained deterministically as the eigenvalue of $\hat{N}_+$, implemented by a standard QND measurement of total occupation number.

E.3 Multiplication as a Deterministic Observable

On the same two-mode space, define $\hat{N}_\times := \hat{N}_A\,\hat{N}_B$. Then $\hat{N}_\times(|m\rangle_A\,|n\rangle_B) = (m\cdot n)\,|m\rangle_A\,|n\rangle_B$. Operationally, $\hat{N}_A\hat{N}_B$ can be measured via an effective cross-Kerr interaction Hamiltonian $\hat{H} = \chi\,\hat{N}_A\hat{N}_B$, where phase accumulation encodes the product mn.

E.4 Verification of Robinson Arithmetic Q (Operator-Level)

On the number basis, the axioms for $+$ and $\times$ reduce to identities of eigenvalues, hence hold exactly on the intended interpretation domain: $x+0 = x$ and $x+Sy = S(x+y)$ hold because $(m+0) = m$ and $(m+(n+1)) = (m+n)+1$. Similarly, $x\cdot 0 = 0$ and $x\cdot Sy = (x\cdot y)+x$ hold because $m\cdot 0 = 0$ and $m\cdot(n+1) = mn+m$. Therefore the interpretation of Q is realized on Fock number states using the observables $\hat{N}_+$, $\hat{N}_\times$, and the successor shift $\hat{V}$.

E.5 Physical Consistency (Energy Conditions and Backreaction)

For the encoding states used in the main text, the relevant excitations can be chosen as number states and restricted orthogonal-sector superpositions as in §3.7. In that regime: (i) $\langle T_{\mu\nu}\rangle$ satisfies the energy-condition statements asserted in the main text for the chosen class of states; (ii) the total excitation energy E_{int} can be made arbitrarily small compared to Mc^2 by choosing suitable occupation numbers and frequency support, so that the backreaction bounds of §3.8 apply.

The successor, addition, and multiplication structures required for diagonalization are implemented explicitly and conventionally in multi-mode QFT, without requiring any nonstandard dynamical "arithmetic machine."

Lemma E.1 (Syntactic internalization of Q). *Let Φ_0 be the recursively axiomatizable fragment of §2.2. Then for each axiom α of Robinson Arithmetic Q and the interpretation ι mapping α to an operator-algebraic statement about $\hat{V}$, $\hat{N}_+$, $\hat{N}_\times$:*

$$\Phi_0 \;\vdash\; \iota(\alpha).$$

Consequently Φ_0 interprets Q in the standard sense of relative interpretability ([33], Ch. 7).

Proof. Each translated axiom $\iota(\alpha)$ is an identity of eigenvalues on the number-state basis (§E.4). These identities are instances of the equational theory of the operator algebra $\mathfrak{A}(\mathrm{DOC}, g)$; the equational theory is included in the deductive apparatus of Φ_0 by clause (ii) of §2.2 (axioms for $\hat{V}$, $\hat{N}_+$, $\hat{N}_\times$ and their eigenvalue relations on the number-state basis). Each $\iota(\alpha)$ is therefore derivable in Φ_0, not merely true in the operator model. ☐

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E.6 Necessity of the Evaluation Map

The evaluation map $\hat{\phi}\colon \mathbb{N} \times \mathbb{N} \to \{0,1\}$, defined by $\hat{\phi}(n,m) = 1$ iff $\Phi_{D_O} \vdash \theta_n(m)$, is not an additional construction imposed on the physical theory: it is the canonical mathematical object forced by Hypothesis (H1) alone. We make this necessity explicit.

Step 1: The theorem set of Φ_0 is Σ_1. The recursively axiomatizable fragment Φ_0 (§2.2) has a recursively enumerable axiom set by construction (components (i)–(v) are each finite or RE). By the Kleene Normal Form theorem [29], the set of theorems of any RE-axiomatized theory is Σ_1: it is semi-decidable by enumerating all proofs. The proof relation $\mathrm{Proof}_{\Phi_0}(p,n)$—"$p$ codes a Φ_0-proof of the sentence with Gödel number n"—is primitive recursive, and the provability predicate $\mathrm{Prov}_{\Phi_0}(n) \equiv \exists p\, \mathrm{Proof}_{\Phi_0}(p,n)$ is Σ_1.

Step 2: Every Σ_1 predicate is representable in Φ_0. Since Φ_0 interprets Robinson Arithmetic Q (Hypothesis (H1) and Lemma E.1), every Σ_1 predicate on $\mathbb{N}$ is representable in Φ_0 by a formula (Representability Theorem; [33], Ch. IV). Applied to Prov_{Φ_0}, this yields a formula $\mathrm{Prov}_{D_O}(x)$ in the language of Φ_0 such that, for each $n \in \mathbb{N}$: if $\Phi_0 \vdash \varphi_n$ then $\Phi_0 \vdash \mathrm{Prov}_{D_O}(\bar{n})$; and if $\Phi_0 \nvdash \varphi_n$ and Φ_0 is Σ_1-sound (Hypothesis (H4)), then $\Phi_0 \nvdash \mathrm{Prov}_{D_O}(\bar{n})$. The evaluation map $\hat{\phi}(n,m)$ is the characteristic function of Prov_{D_O} applied to the Gödel number of $\theta_n(m)$.

Step 3: The evaluation map is canonical and unique. Any two Σ_1 formulas representing the same RE set are provably equivalent in Q ([33], Ch. IV, Corollary 4.3). Hence the evaluation map is uniquely determined, up to provable equivalence, by the axiomatic content of Φ_0. Different choices of Gödel numbering or coding scheme yield recursively equivalent evaluation maps (Proposition G.3). The evaluation map is therefore *not a construction added to the physical theory*: it is the unique canonical Σ_1 representation forced by Φ_0 being recursively axiomatizable and interpreting Q.

Remark E.2 (What this implies for the Lawvere argument). Proposition 8.4 constructs the physical Lawvere instance from $\hat{\phi}$. Steps 1–3 show that $\hat{\phi}$ is not a *choice* but a *consequence*: it is the canonical Σ_1 proof predicate forced by (H1). The non-surjectivity of $\hat{\phi}$ required by the Lawvere contrapositive is equivalent to the Gödelian incompleteness of Φ_{D_O}—a consequence of (H1)–(H4), not an additional assumption. The diagonal obstruction is therefore as necessary as the hypotheses of the Master Theorem themselves.

E.7 Explicit Encoding Example (Toy Model)

We illustrate the encoding map ι with a concrete example using a two-mode interior Fock space $\mathcal{H}_A \otimes \mathcal{H}_B$.

Target. Encode the natural number $n = 12$ at resolution $N = 2$, with observables $O_1 = \hat{N}_A$ (mode-A occupation) and $O_2 = \hat{N}_B$ (mode-B occupation), and discretization $Q_i(x) = \lfloor x \rfloor$.

Decoding. From the prime-power coding $\gamma_2([s]) = 2^{\mathrm{enc}(q_1)} \cdot 3^{\mathrm{enc}(q_2)}$, we decode $12 = 2^2 \cdot 3^1$, yielding $\mathrm{enc}(q_1) = 2$ and $\mathrm{enc}(q_2) = 1$. Under the encoding convention $\mathrm{enc}(k) = \langle +1, k, 1\rangle$ for non-negative integers, this gives $q_1 = 2$ and $q_2 = 1$.

State construction. Choose the interior state $|\psi_{\mathrm{int}}\rangle = |2\rangle_A \otimes |1\rangle_B$. Then $\langle \hat{N}_A\rangle = 2$ and $\langle \hat{N}_B\rangle = 1$, so $Q_1(\langle O_1\rangle) = 2$ and $Q_2(\langle O_2\rangle) = 1$, yielding $\gamma_2([\psi_{\mathrm{int}}]) = 12$ as required.

Admissibility. The state $|\psi_{\mathrm{int}}\rangle$ is a finite-particle excitation of the Fock vacuum (hence Hadamard), with total energy $E_{\mathrm{int}} = 2\hbar\omega_A + \hbar\omega_B$. For $\omega_{A,B} \sim c/r_{\mathrm{s}}$ and $M = M_\odot$, this gives $E_{\mathrm{int}}/Mc^2 \sim 10^{-76}$, satisfying the backreaction bound.

General procedure and computability. For an arbitrary code n, the inverse encoding proceeds as follows: (1) compute the prime factorization of n (primitive recursive); (2) decode each exponent via enc^{-1} to recover the rational target tuple $(q_1, \ldots, q_N)$ (primitive recursive); (3) for integer targets $q_i \in \mathbb{N}$, set the interior state to the product number state $|q_1\rangle \otimes \cdots \otimes |q_N\rangle$; for rational non-integer targets, solve the finite system $\langle\psi|O_i|\psi\rangle = q_i$ over a truncated Fock basis (a finite-dimensional linear algebra problem, computable by Gaussian elimination over $\mathbb{Q}$). The split property ensures that this interior modification does not affect any exterior observable. The resulting map $\iota\colon n \mapsto s_n$ is therefore *Turing-computable*: given n as input, a Turing machine can output a finite specification of the corresponding interior state s_n (mode occupation numbers or, for superpositions, a finite list of amplitudes). This is the effective encoding required for the diagonal construction of Section 4.1.

F Notation Guide

This appendix consolidates the notation introduced throughout the paper. A more detailed list, with definitions, is given in §2.1; the present appendix provides a quick-reference summary organized by domain.

Geometry. (M, g): spacetime manifold and metric. $\mathscr{I}^\pm$: future/past null infinity. $i^\pm$: future/past time-like infinity. $J^\pm(p)$: causal future/past of p. $\mathcal{H}$: event horizon, $\mathcal{H} = \partial J^-(\mathscr{I}^+)$. r_{s}: Schwarzschild radius, $r_{\mathrm{s}} = 2GM/c^2$.

Quantum theory. $\mathcal{H}$: Hilbert space. $|\psi\rangle$: quantum state vector. $[\psi]$: gauge-equivalence class. $\hat{O}$: observable (operator). Obs_O: observables accessible to observer O.

Logic. Φ_{semi}: semiclassical framework. Φ_{ext}: exterior-restricted theory. $\vdash$: derivability (provability). $\models$: semantic entailment. $\mathrm{Desc}(\Phi, s)$: complete description predicate. $\ulcorner\varphi\urcorner$: Gödel number of formula φ. γ: gauge-invariant Gödel encoding.

Thermodynamics. S_{BH}: Bekenstein–Hawking entropy. T_H: Hawking temperature. A: horizon area. k_{B}, $\hbar$, c, G: standard constants.

G Categorical Details (Triadic Correspondence)

This appendix collects the technical statements organizing the triadic correspondence of §2.8. None of the material here is logically required for the incompleteness theorems of Sections 4 and 4.12; it makes precise the categorical scaffolding on which the structural correspondence is built.

G.1 Detailed Object and Morphism Specifications

$\mathbf{Cat}_G$ — full specification. Objects: domains of outer communication (DOC) of stationary, asymptotically flat, electrovac black hole spacetimes, considered up to isometry. For a black hole spacetime (M, g) with future null infinity $\mathscr{I}^+$ and event horizon $\mathcal{H} = \partial J^-(\mathscr{I}^+)$, the DOC is

$$\mathrm{DOC} := J^+(\mathscr{I}^-) \cap J^-(\mathscr{I}^+),$$

i.e., the maximal region that can both receive signals from $\mathscr{I}^-$ and send signals to $\mathscr{I}^+$. In the standard uniqueness setting (stationarity, asymptotic flatness, electrovac, and the usual regularity hypotheses), the exterior geometry is characterized by Kerr–Newman parameters (M, J, Q) up to isometry. Morphisms are causal isometric embeddings $\psi\colon \mathrm{DOC}_1 \to \mathrm{DOC}_2$ preserving the relevant asymptotic structure. Interior regions are not part of $\mathbf{Cat}_G$ objects; interior degrees of freedom enter indirectly via the inability of an exterior-restricted theory to determine states beyond the horizon.

$\mathbf{Cat}_P$ — full specification. Objects are locally covariant quantum field theories on globally hyperbolic spacetimes in the BFV sense [37, 46]: each (DOC, g) in $\mathbf{Cat}_G$ is assigned a unital $*$-algebra $\mathfrak{A}(\mathrm{DOC}, g)$ satisfying isotony, covariance, microcausality, and the time-slice property. For stationary black hole exteriors, $\mathfrak{A}(\mathrm{DOC}, g)$ contains an asymptotic subalgebra $\mathfrak{A}_\infty$ generated by gauge-invariant asymptotic charges (ADM mass, angular momentum, total electric charge). Morphisms are injective $*$-homomorphisms $\alpha_\psi\colon \mathfrak{A}(\mathrm{DOC}_1, g_1) \to \mathfrak{A}(\mathrm{DOC}_2, g_2)$ induced by causal isometric embeddings, as required by local covariance.

$\mathbf{Cat}_L$ — full specification. Objects are consistent, recursively axiomatizable formal systems $(\mathcal{L}, \vdash)$ that interpret Robinson Arithmetic Q, considered up to bi-interpretability (or recursive equivalence of Gödel numberings). $\mathbf{Cat}_L$ is restricted to the physically realized subcategory arising as the essential image of the Gödel-encoding functor G defined below. Morphisms are bi-interpretations preserving provability in both directions, up to the equivalence notion fixed in the framework.

G.2 Functor F: Locally Covariant Quantization

On objects: $F(\mathrm{DOC}, g) := \mathfrak{A}(\mathrm{DOC}, g)$. On morphisms: for each causal isometric embedding $\psi\colon (\mathrm{DOC}_1, g_1) \to (\mathrm{DOC}_2, g_2)$, local covariance provides an injective $*$-homomorphism $F(\psi) := \alpha_\psi\colon \mathfrak{A}(\mathrm{DOC}_1, g_1) \to \mathfrak{A}(\mathrm{DOC}_2, g_2)$.

In asymptotically flat spacetimes the ADM mass and related charges can be expressed as boundary integrals at spatial infinity:

$$M_{\mathrm{ADM}} = \frac{1}{16\pi G} \lim_{r\to\infty} \oint \left(\partial_j h_{ij} - \partial_i h_{jj}\right) dS^i\,, \qquad (27)$$

with $h_{ij} = g_{ij} - \delta_{ij}$ in asymptotically Cartesian coordinates, and similarly for angular momentum and electric charge via standard Komar/Gauss expressions. These charges are gauge-invariant elements of $\mathfrak{A}_\infty$.

Remark G.1 (F does not depend on a choice of vacuum). In the algebraic/BFV framework the algebra $\mathfrak{A}(\mathrm{DOC}, g)$ is defined independently of state: vacuum choices (Unruh, Hartle–Hawking, Boulware, etc.) enter only when selecting a state on $\mathfrak{A}(\mathrm{DOC}, g)$, not when defining the algebra itself. Different quantization prescriptions for the same classical field content can in general lead to non-isomorphic functors; the present paper assumes a fixed physically chosen theory throughout.

G.3 Functor G: Gödel Encoding and Coding Invariance

Fix a finite-resolution level N and a gauge-invariant coding scheme as in §2.4: choose a countable family $\{O_i\}$ of gauge-invariant observables and rational quantizers $\{Q_i\}$, and define the finite coding datum $C_N(s)$. From $\big(\mathfrak{A}(\mathrm{DOC}, g), \text{coding scheme}\big)$ we build a recursively axiomatizable formal system $\mathcal{L}_{\mathfrak{A},N}$ whose intended semantics is given by coded expectation values of the selected observables in admissible states, together with the deductive apparatus required for the diagonal arguments (an interpretation of Q plus a provability predicate for the chosen axiom set).

On objects: $G(\mathfrak{A}(\mathrm{DOC}, g)) := \mathcal{L}_{\mathfrak{A},N}$. On morphisms: a $*$-homomorphism $\alpha\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ induces a translation τ_α between the corresponding coded predicate languages (up to the chosen coding equivalence), yielding a morphism in $\mathbf{Cat}_L$.

Remark G.2 (Injectivity of the encoding). Injectivity of the encoding is claimed only at the appropriate level: the coding is injective on finite-resolution equivalence classes $\mathcal{S}/{\sim}^N_{\mathrm{phys}}$ (§2.4), not on the full uncountable state space.

Proposition G.3 (Invariance of G under choice of effective coding). *Fix a finite-resolution coding level N and a* *class of acceptable effective codings. Any two such codings (and the induced formal systems) are recursively equivalent: there exist total computable translations between the corresponding code sets, and the associated derivability/undecidability statements are invariant under the choice of coding.*

Proof sketch. This is the standard "invariance under Gödel numbering" principle: any two effective codings of the same finite syntactic or physical description data are related by total computable translations [29]. The diagonal constructions and incompleteness statements therefore do not depend on which acceptable coding is chosen, provided the coding is effective and injective on the chosen finite-resolution equivalence classes. □

G.4 Functor H: Geometric Realization

We define H only on the essential image of $G \circ F$ (the physically realized subcategory of $\mathbf{Cat}_L$). Given $\mathcal{L}$ in that image, we extract coarse exterior parameters (M, J, Q) from the coded asymptotic content at the fixed resolution and reconstruct the corresponding Kerr–Newman exterior:

$$H(\mathcal{L}) := \mathrm{DOC}\big(\mathrm{KN}(M, J, Q)\big),$$

where $\mathrm{KN}(M, J, Q)$ denotes the Kerr–Newman solution with those parameters under the standard uniqueness hypotheses.

Remark G.4 (Interior encoding). When needed for the incompleteness construction, we additionally specify a physically realizable interior encoding of a given code (or fixed-point sentence) using interior field degrees of freedom while keeping backreaction parametrically small. This "interior realization" is part of the physical construction of §3–4; it is not an object of $\mathbf{Cat}_G$, whose objects are exteriors/DOCs.

Proposition G.5 (H is a canonical choice, not forced uniquely)**.** *Given a coded exterior parameter triple (M, J, Q) extracted at a fixed decoding resolution, one may choose a canonical exterior geometry by selecting the Kerr–Newman solution with those parameters under the standard uniqueness hypotheses. This provides a canonical representative in $\mathbf{Cat}_G$ for organizing the correspondence on the stationary subfamily. Beyond the stationary/uniqueness regime, however, there are many inequivalent geometric realizations compatible with the same coarse exterior data.*

Remark G.6 (Minimal-mass convention). If one wishes to define a "minimal" stationary carrier geometry for an encoding of size $n_{\max}$, one may impose the entropy-capacity requirement $S_{\mathrm{BH}}/k_{\mathrm{B}} \geq \ln(n_{\max})$, which for Schwarzschild gives a scale

$$M_{\min} \gtrsim \sqrt{\frac{\hbar\, c\, \ln(n_{\max})}{4\pi G}}.$$

This is a convention selecting a minimal-mass Schwarzschild representative; it is not a uniqueness theorem, but a canonical choice within a restricted family.

G.5 Triadic Retraction: Proof of Proposition 2.16

Proof of Proposition 2.16. Direction $H \circ G \circ F$. For (DOC, g) a Kerr–Newman exterior with parameters (M, J, Q), $F(\mathrm{DOC}, g) = \mathfrak{A}(\mathrm{DOC}, g)$ contains $\mathfrak{A}_\infty$ with ADM mass, angular momentum, and electric charge as gauge-invariant elements (see (27)). The coding scheme of §2.4, applied to a coding family containing these asymptotic charges (which we may assume without loss of generality by Proposition G.3), encodes (M, J, Q) at the coarse decoding resolution. Then H recovers the Kerr–Newman exterior with the encoded parameters, which by Lemma G.7 (parameter invariants under isometries) is isometric to the original (DOC, g).

Direction $G \circ F \circ H$. Given $\mathcal{L}$ in the essential image of $G \circ F$, $H(\mathcal{L})$ produces the canonical Kerr–Newman exterior, whose BFV algebra has the same coded asymptotic content as $\mathcal{L}$ at the chosen resolution. The associated formal system $G(F(H(\mathcal{L})))$ is therefore bi-interpretable with $\mathcal{L}$ at that resolution by Lemma G.8 (naturality of decoding) and Proposition G.3.

In both directions the equivalence is up to the explicitly stated notions: isometry/diffeomorphism for $\mathbf{Cat}_G$, $*$-isomorphism (or local covariance equivalence) for $\mathbf{Cat}_P$, and recursive equivalence/bi-interpretability for $\mathbf{Cat}_L$. □

Lemma G.7 (Parameter invariants under isometries of Kerr–Newman)**.** *Let $(M_1, g_1) = \mathrm{KN}(M_1, J_1, Q_1)$ and $(M_2, g_2) = \mathrm{KN}(M_2, J_2, Q_2)$ be Kerr–Newman spacetimes, and let ψ be an asymptotically flat isometry between them. Then ψ preserves the ADM mass, Komar angular momentum, and total charge; consequently, $(M_1, J_1, Q_1) = (M_2, J_2, Q_2)$.*

Proof sketch. ADM mass and charge are asymptotic invariants; Komar integrals for stationary axisymmetric solutions are preserved under asymptotically flat isometries. □

Lemma G.8 (Naturality of decoding under recursively equivalent encodings)**.** *Let γ and γ' be two acceptable (recursively equivalent) gauge-invariant encodings of the same physical equivalence classes, and let D be any decoding functional extracting discretized observable data from the encoding. Then $D \circ \gamma$ and $D \circ \gamma'$ agree up to the fixed discretization scheme.*

Proof sketch. Recursively equivalent encodings differ by a computable translation on codes; when decoding is defined by the same underlying discretized observable data, the induced equivalence relation is invariant under such translations. □

Remark G.9 (Scope of functorial claims). Functorial statements in §2.8 are to be understood at the level of (i) gauge-invariant observable algebras, (ii) their state spaces modulo the physical equivalence relation defined by discretized separating families, and (iii) the corresponding effective logical encodings. Claims of strict functoriality beyond this operational level are not required for any incompleteness result proved in the main text.

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